Overview of Inverse Problems

Introduction

In inverse problems the goal is to find objects, sources, or changes in medium properties from indirectly related data. The solution is usually given as an image and as such the word imaging is often a descriptor for an inverse problem. What distinguishes an inverse problem from, e.g. an image deblurring problem, is that the mathematical model, for the process that generates the data, plays an integral role in the solution of the problem.

Inverse problems are ill-posed. This means that: (1) there may be more than one solution; or (2) small changes in the data may yield large changes in the solution; or (3) a solution may not always exist, especially when the data is noisy. To improve the likelihood of a unique stable solution, several approaches are taken: (1) one is to add more data by doing a sequence of very similar experiments or by measuring additional properties of the output of an experiment; this improves the likelihood of a unique, sometimes stable, solution; when the data is noisy a solution may not exist; however an approximate solution may exist; (2) a second possibility is to treat the problem as an optimization problem adding a regularizing term to both improve the mathematical properties of the cost functional, that is to be optimized, and to add mathematical structure to find the approximation properties of the solution; (3) a third possibility is to reduce the sought after properties of a solution; e.g. instead of looking for all the changes in a medium one might seek only the support of the region where changes occur or one might look only for the discontinuities of the medium; and (4) a fourth possibility is to use coupled physics, that is to utilize the two physical properties simultaneously; an example of this is elastography where one mechanically creates shear motion in the tissue while simultaneously taking a sequence of RFQ (ultrasound) or a sequence of MR (Magnetic Resonance), data sets, see [S4] or, for more examples, the Coupled Physics special issue of Inverse Problems, 28(8), 2012.

An important feature in inverse problems is to utilize a realistic mathematical model whose numerical or exact solution can be shown to be consistent with measured data and to use the model to make the correct physical interpretation of the inverse problems solution. The
mathematical structure utilized to obtain the solution is also related to the targeted solution feature. For example, (1) in the viscoelastic and wave equation models [L3] the arrival time of waves together with the Eikonal equation, can be utilized to find the fastest compression, shear or acoustic wavespeed; (2) microlocal analysis can be applied to a broad set of models and is well suited for finding discontinuities with essentially far field data; (3) linear sampling and factorization methods yield the support of inhomogeneities in a constant background, again with far-field data; and (4) in media with well distributed small scale fluctuations and most accurately modeled as random media, the inverse problem is a source location problem and cannot be an inhomogeneity identification problem.

**1-D Inverse Spectral Problems**

A first question to ask is: Given: (1) the form of a mathematical model, that is a second order eigenvalue problem with square integrable potential and Dirichlet boundary conditions on a finite interval; and (2) the eigenvalues for this problem, does there exist a unique potential corresponding to the mathematical model and the eigenvalues? The answer is yes provided that the eigenvalues satisfy a suitably general asymptotic form and the potential is symmetric on the interval [G1, H4, P1].

If one relaxes the condition on the potential, that is, it is no longer symmetric, then one must add additional data to obtain existence and uniqueness of the solution. Here there are essentially three choices: (1) for each eigenfunction, add the ratio of the first derivative at an endpoint to the $L^2$ norm of the eigenfunction [G1]; or (2) for each eigenfunction add the ratio of the first derivatives at the end points for each of the eigenfunctions [P1]; or (3) add a second sequence of eigenvalues where one of the boundary conditions is changed and has an unknown coefficient to be determined by the data [L2]. Each of these additional sequences must satisfy a suitably general asymptotic form, as before.

If one relaxes smoothness on the potential or equivalently changes the second order differential equation to have a positive impedance with square integrable derivative, see [C1, C2], similar statements as in (2) above apply with the exception that the data sequences have weaker asymptotic forms and with Dirichlet boundary conditions one additional data must be added, e.g. the integral of the impedance over the interval where the problem is defined. If the smoothness is further relaxed so that, e.g., the positive impedance is of bounded variation, the asymptotics that can be established for the eigenvalues has very limited structure, see [H1, H5]. Here some progress has been made with the boundary control method, see [S3] but there are still many open problems. See also [S3] for more discussion of 1D inverse spectral problems and also [M2].

**Inverse Nodal Problems**
In 1D a related but very different problem to that described above is to use the nodes, or zeros, of eigenfunctions as data for the inverse problem. These are points in a vibrating system where there is no vibration. This inverse problem was first defined in [M1] for a square integrable potential and was later extended to an impedance with integrable first derivative and densities in \( \text{BV} \) \([H2, H1]\). Algorithms based on this idea have remarkable convergence properties, even under the weakest conditions on the unknown coefficients \([H1, H2]\). Extensive stability estimates in the case where the potential is square integrable or smoother are described in [L1].

Inverse Nodal problems in 2D are significantly more difficult partly due to the fact that the detailed asymptotics that can be obtained in 1D is much more difficult to obtain in 2D. Nevertheless, for a rectangular domain and with square integrable potential, a fundamental result is obtained in [H3, M6] that established uniqueness and an algorithm for finding the potential from nodal lines.

**Elastography: A coupled physics imaging modality**

The goal in elastography is to create images of biomechanical properties of tissue. It is inspired by: (1) the doctors’ palpation exam where the doctor presses against the skin to feel abnormalities in the body; and (2) compression and vibration experiments that show cancerous tumors have shear wave speed with more than twice the value in normal tissue, \([T2]\), or similar experiments that show fibrotic and cirrhotic liver can have shear moduli at least double that of normal liver tissue, \([Y1]\).

This capability, i.e. the imaging of biomechanical properties of tissue, is created by combining two physical properties of tissue. The basic idea is that the tissue is mechanically moved in a shearing motion and while it is moving a sequence of RFQ ultrasound data sets are acquired or a sequence of Magnetic Resonance, MR, data sets are acquired. The sequences are processed to produce a movie of the tissue moving within the targeted area of the body. The moving displacement data sets are the data for the inverse problem. The success of these experiments is based on the fact that tissue is mostly water. This means that the ultrasound data acquisition utilizes the fact that compression waves in the body travel at nearly 1500 m/s while shear waves, which are the mechanically induced motion, travel at approximately 3 m/sec in normal tissue. So the shear wave can be considered as fixed during a single ultrasound sequence data acquisition. The compression and the shear waves are the two physical properties that are coupled for this type of experiment. The MR acquired data sets are based on the spin of water molecules, a property distinct from the shear wave propagation; MR and shear wave propagation are thus the two physical properties that are coupled for this experiment. These coupled physics imaging data acquisition modalities are also referred to as hybrid imaging modalities.
The mechanical motion is induced in one of three ways: (1) the tissue is moved with a sinusoidal motion at a rate from 60-300Hz; both MR and RFQ data sets can produce the targeted movie; e.g. this is done with RFQ data sets in sonoelasticity, [P2], or with MR data sets in MRE, [M7], [M8]; (2) motion is induced by an interior radiation force push which is a pulse induced within the tissue by focused ultrasound, [S6]; excellent examples of the use of this method are supersonic imaging [B2], and ARFI and SWEI, [N1, N2]; and (3) the tissue is compressed in very small increments and is allowed to relax between compressions; see [O1] for the initial compression experimental results.

When the tissue is mechanically excited by a pulse, a wave propagates away from the pulse location. This wave has a front, a moving surface where ahead of the front there is very little motion and at the front there is large amplitude. The front has a finite propagation speed. The time, T, at which the front arrives at a given location, the arrival time, is the richest data subset in the movie dataset. Under suitable hypotheses, it can be shown that T is the viscosity solution of an Eikonal equation, $|\text{grad } T|^2 = \left(\frac{1}{c}\right)^2$, where c is the shear wave speed; this property is utilized by the arrival time algorithm, [M9],[M10],[M11]. The quantity, c, is the imaging functional. Another algorithm, the time-of-flight algorithm utilizes a one-dimensional version of this Eikonal equation, see [N1], [B2], for applications in Supersonic Imaging or ARFI or SWEI.

Note that the presence of viscoelasticity can be observed in the time trace at each pixel in the tissue as the pulse spreads in the direction behind the front as the pixel location is taken further and further from the initial pulse location. This occurs because the speed of the frequency content in the pulse propagates at a frequency-dependent wave speed with the fastest speeds at the highest frequencies. A mathematical model that contains both the viscoelastic effect and the finite propagation speed property is the generalized linear solid model. See [M4] for theoretical results for the forward problem, including the finite propagation speed property, and [L3] for the use of this model to create shear wave speed images with Acoustic Radiation Force Induced (CAWE) crawling wave data.

When the tissue is mechanically moved in a sinusoidal motion, a time harmonic wave is induced. The amplitude of this wave satisfies the time harmonic form of the linear elastic system and is sometimes simplified to the Helmholtz equation. A viscoelastic model is required and in the time harmonic case, this yields complex valued coefficients. The Helmholtz or elastic model, as opposed to the Eikonal equation, for the frequency dependent displacement is utilized. The most often used imaging functional is the complex valued shear modulus. The Fourier Transform of the general linearized solid model is appropriate here. For this model the complex valued shear modulus approaches a finite limit as the frequency goes to infinity. Other viscoelastic models have been utilized, see [J1], [B9], [Z2]; in [J1] and [B9] the complex valued shear becomes unbounded as the frequency becomes large; at the same time, the application of the model in [J1] is usually for fixed frequency, but not in [B9]. For fixed frequency, the shear modulus satisfies a
first order partial differential equation system. The difficulties here are that: (1) the solution, that is the shear modulus, is complex valued so some known methods, e.g. computing along characteristic curves cannot be employed; and (2) the coefficients of the first derivative terms can be zero on a large set of points, lines or surfaces (but not in open subsets of the region of interest). This makes stability and uniqueness results difficult to obtain; however, uniqueness and stability results are contained in [H6] where solutions and coefficients are assumed to be subanalytic; an earlier paper, [A1], achieved results when: (1) the frequency is zero, (2) the dimension is 2, and (3) the shear modulus is real: under more general smoothness conditions.

A number of approaches are utilized to overcome this difficulty of having possible zero values of the coefficients of the first derivative terms: (1) set all derivatives of the shear modulus to zero and solve the resulting problem [S4]; a bound on the error that is made when this is done, under the assumptions that the coefficients of the first derivative terms are nonzero, is contained in [L4] when the coefficients are real; the same proof will yield a similar result when coefficients are complex valued; (2) another is to first linearize the problem about a base problem, then use multiple data sets to eliminate the problem of having zero coefficients of the derivatives of the sought after shear modulus; see, for example, [B5]; and (3) employ optimization and iterative, [J1], methods.

**Inverse Scattering Problems**

The first type of inverse scattering problem is defined as follows. Suppose: (1) a constant, infinite in extent, background surrounds a bounded object or a bounded region containing an inhomogeneous medium; (2) one initiates an incoming plane harmonic wave that scatters from the object or region; and (3) the scattered wave is measured in the far field. Then the classic acoustic, electromagnetic or elastic inverse scattering problem is to recover the object or inhomogeneity from the scattered data. Even when the forward problem is linear, the inverse problem is nonlinear. First the mathematical structure for the forward problem must be established and then one can address the inverse problem. Considerable mathematical structure has been developed toward solving the inverse problem and the literature is vast. It has been shown, in the acoustic and electromagnetic case, that if one has scattering data at all outgoing angles from an incoming plane wave from a single direction and oscillating at a single frequency then only one polygonal object can correspond to that data [L7, L8, L9]; for the inhomogeneous medium problem, it is known that if one has scattering data at all outgoing angles, for incoming waves from all incoming directions and oscillating at a single frequency, then an inhomogeneous medium in a bounded region is unique, [N4], [H8], [B10], [C3]. Nevertheless the problem is severely ill-posed. The ill-posedness arises for several reasons, one being that the information content lies deep in the data and another being that some operators developed in the solution structure have only dense range. The latter means that solution methods can be unstable in the presence of noise in the data or when approximations of the data are used.
To cope with the ill-posedness, several approaches are taken; here are a few examples: (1) the problem can be linearized; this is called the Born Approximation; this doesn’t remove the ill-posedness but the ill-posedness of the linearized problem is somewhat easier to analyze; (2) one can add data by including scattered waves from incoming waves, both from a full set of possible directions, and in addition there are incoming waves, together with their scattered waves, with many frequencies of oscillation; this can be accomplished, as in [B11] for the inverse medium problem where the Born approximation is used for each frequency of oscillation, and one iterates by solving the linearized problem for one frequency, using the output from the solution of that linearized problem as input for solving the linearized problem for another frequency, and so on a similar method is applied in [B12] for the inverse source problem; and/or (3) one can target a simpler property, e.g., find only the support of a bounded inhomogeneous region; this approach is taken in the Linear Sampling method, [C3], [C4],[P3], and in the Factorization method, [L6], [K5], where the problem is reformulated so that the boundary of the region or object is identified as the points where an imaging functional becomes large.

The second type of inverse scattering we consider is scattering of a wave in a half space (for example, a section of the earth relatively near the surface) from a source, for example a buried explosive or an earthquake. Here it is often assumed that a smooth, slowly varying in space, background medium is known and what is sought are the abrupt changes, usually referred to as discontinuities and also described as the highly oscillatory part of the medium. What is remarkable here is that in [B13] it was shown, under certain assumptions, that the measured pressure field at locations on the surface could be expressed as integrals, referred to as transforms, over space-time surfaces. The integrand of these integrals contains the sought after unknown in the half space plus possibly some known quantities. When this transform is a Fourier Integral Operator (FIO), using concepts from microlocal analysis, see [S1, S5, N2] an inversion, containing two steps referred to as a migration step together with a microlocal filter, to recover the highly oscillatory part is possible, [see N2 and the references therein].

A third type of inverse scattering is more related to a method for solution than a specific physical setting and is referred to as an Adjoint Method [N3]. The main feature is that the method makes use of the adjoint of the derivative of the forward map. An example is the iterative method known as the Kaczmarz method. We describe the method where there are a finite number of sources and a corresponding number of responses, measured for example on the boundary of the medium to be imaged. An initial guess is made for the medium. At each update, the adjoint of the derivative of the forward operator is computed for the current value of the medium. The new approximation of the medium is obtained by having that adjoint operate on the difference between the simulated ‘data’ computed with the current iterate minus the measured data due to the next source in the iteration. See [N3] for further discussion. The method has been applied to a wide range of problems including optical, impedance, ultrasound tomography and computerized tomography (CT).
Computerized Tomography

The inversion of the second inverse scattering problem described above has similarities with the inversion of the Radon Transform utilized in X-Ray tomography. X-Ray CT has a rich history with Cormack [C7] and Hounsfield sharing a 1979 Nobel Prize for the discovery and initial practical implementation. The mathematical problem is the recovery of the X-Ray attenuation (often referred to as density) from line integrals of the attenuation. This problem was first solved by Radon, [R2]; the integrals over lines or planes of an integrable function is now referred to as the Radon transform. There is a vast literature on inversion formulas to find the pointwise attenuation from line integrals of the attenuation. Of particular interest are: (1) inversion formulas that utilize Fourier multiplication operators, such as are derived from the Hilbert transform, and also the formal adjoint of the Radon transform; regularization can be applied to avoid mild illposedness that can occur with the use of such formulas, see[F2]; (2) formulas for data taken in 3D on helical curves; this is motivated by the use of a circular scanner that takes data as the patient is moved linearly through the scanner; remarkably Katsevich [K6] discovered an exact formula when the attenuation is smooth and when it is assumed that the helix lies on a cylinder; additional very useful formulas were obtained in [G2], [Z1] which relate the derivative of the line integral transform to the Hilbert transform; and (3) inversion formulas that utilize an expansion of the unknown attenuation in terms of a set of basis functions; of particular interest are basis functions known as ‘blobs’ which have representations in terms of Bessel functions; once a representation is chosen, then the inversion is formulated as an iteration of algebraic reconstructions, see the discussion of ART in [H7].

Time Reversal and Random Media

Time reversal invariance in acoustic, elastic, and electromagnetic wave propagation is the basic concept behind time reversal mirrors. In free space, in an enclosed region, if a source emits a signal and: (1) the signal is measured for a very long time, at all points on the boundary; and (2) the received signal on the boundary is time reversed and back propagated into the region: then the back propagated signal will focus at the original source location.

When the medium contains very well distributed scatterers of varying small sizes, it is not possible to image the inhomogeneities. What is remarkable is that, in this case, when those inhomogeneities can be thought of as being randomly distributed with a not too large variance, then the multiple scattering from the inhomogeneities provides a significant advantage in locating the source. Indeed what occurs is that: (1) limited aperture measured signals, when back propagated, can have excellent refocusing properties; (2) the refocusing breaks the diffraction limit; (3) even in unbounded domains, the back propagated signals focus; and (4) the length of the measured received signals can be quite short.
All of these properties have been well demonstrated experimentally by Mathias Fink’s group, see [F1].

This is very much related to coherent interferometric imaging, see [B3], where the fundamental tool is local cross correlations of the data traces at nearby receivers, estimates of the frequency range for which wave fields are statistically correlated and favorable resolution limits can be obtained. See also [B1] for additional descriptions of the advantages of cross correlations when dealing with random media.

As multiple scattering is the main physical property that is yielding the advantage here, additional work has been done in waveguides containing random media when additional multiple scattering from the boundaries provides even more advantage, see [A7]. In contrast, see [D1] for recovery of inhomogeneities in waveguides that do not contain random media.

**Optical Tomography**

Optical tomography is an imaging modality that uses the transmission and reception of light at collectors to image properties of tissue. Light scatters significantly in tissue making the imaging problem quite difficult and ill-posed. Furthermore, the mathematical formulation of the inverse problem depends on the scale of the scattering that is taken into account; the mathematical formulation depends fundamentally on the experimental setup. The latter is well described in [A4]. Mathematical formulations can utilize the radiative transport equation or the diffusion equation; both formulations are discussed in [S8].

**Hybrid, or Coupled Physics, Based Imaging Modalities**

As medical imaging and nondestructive testing have become widely used, a broad set of experiments to provide useful data have been defined. Initially, a medium, e.g. tissue, was probed by a signal from outside the medium and the scattering from that signal, that is also measured outside of the medium, was utilized to obtain the image or the information needed for nondestructive testing. Typically, the edges of changes in the medium were the property that could most easily be imaged. In many cases these edges could be imaged with high resolution. What was needed were new methods to determine changes within the regions which are defined by the edges or surfaces. This need is being addressed by coupled physics, or also called hybrid, modalities. In these modalities two physical properties of the medium are utilized; e.g. (1) in elastography (see above paragraph) one uses sequences of RFQ ultrasound data, or sequences of MR data, the latter acquired while making a shearing motion in the tissue so that compression wave and water molecule spin properties of the medium are utilized; these ideas are incorporated in ultrasound and MR machines; (2) in photoacoustic imaging [K2], where low frequency electromagnetic (EM) waves expose small regions of the medium to a short pulse, an acoustic wave is emitted and recorded outside the object and from this data the electromagnetic
absorption for the small region is determined; (3) ultrasound modulated optical tomography or ultrasound modulated electrical impedance tomography combine ultrasound which is used to perturb the medium with optical tomography or electrical impedance measurements, see [M5]; and (4) magnetic resonance electric impedance tomography where MR is combined with electric impedance tomography, see [S2]. See also [A3] for additional coupled physics problems and results.

**Inverse Boundary Problems and Inverse Source Problems**

An example of the inverse boundary problem is: (1) posed on a bounded domain; (2) has an electromagnetic model for the forward problem; and (3) has application where one applies, e.g., all possible voltages at the boundary and for each voltage distribution, one measures the current on the boundary. This set of experiments gives data pairs (voltage and current) that define the Dirichlet to Neumann (DtN) map. A similar problem can be defined when the model is the acoustic or elastic model. The goal then is to determine unknown electric, acoustic or elastic properties from the DtN map. A significant literature has built up studying uniqueness, stability, isotropic and anisotropic models and the partial data problem.

We begin with those mathematical models that are elliptic partial differential equations that are defined in the (time) frequency domain. Historically the problem was posed by Calderón [C6] in the isotropic electrostatic case, that is, the frequency is zero. A powerful tool that has been used to study, even the more general frequency not equal to zero problem, and including acoustic, electromagnetic, and elastic models, is microlocal analysis.

We address the uniqueness problem in the acoustic and electromagnetic problem. For this case uniqueness for dimension \( n \geq 3 \) has been established for positive conductivities that are somewhat less smooth than twice continuously differentiable but not as rough as Lipschitz continuous. For dimension two, uniqueness is established for essentially bounded positive conductivities. See [U1] for a discussion of uniqueness and a method for finding the support of an obstacle for the electromagnetic case.

One of the difficulties in using the DtN map in applications is that while the stability for recovering the conductivity from the data has been established, that stability is logarithmic and is quite weak, see [A5].

Nevertheless, partial data, that is knowledge of the DtN map on only part of the boundary can also yield uniqueness for \( n \geq 3 \). The unique recovery of anisotropic properties, properties that depend on direction, from the DtN is not possible as a change of variables that leaves the boundary and the boundary data fixed produces a counterexample to uniqueness. A major question is whether or not this change of variables property is the only obstruction to uniqueness. Toward this end, it has been shown that in dimension two this is the only obstruction to uniqueness under very general smoothness conditions [A6].
A related problem is an inverse source problem where Cauchy data, that is a single Dirichlet, Neumann boundary data pair, is known for a second order, static, linear problem with known coefficients and an unknown source in a bounded domain in \( n \) dimensions. From that data one seeks to determine the source. There are a number of examples to show that the solution is not unique so a typical redefinition of the problem is to look for the minimum, square integrable source. Alternatively one can consider finding sources of constant value but confined to a subregion. Here to obtain uniqueness one makes assumptions on the geometry of the region. The problem is very ill posed with, as in the inverse DTN map problem, logarithmic stability.

If the mathematical model for the physical problem contains a frequency term, as in the Helmholtz equation, or models time dependent waves, with time dependent data, this substantively changes the problem, see [I1]. In this case, use of continuation principles has played a significant role in uniqueness results.

**Distributions, Fourier Transforms and Microlocal methods**

Microlocal methods are an important mathematical tool. The description of these methods requires an excellent understanding of Distributions and Fourier Transforms; see [S1] for this much needed background. As explained in [S5] microlocal analysis is, roughly speaking, the local study of functions near points and including directions.

These methods have been important in the study of inverse problems, particularly in the location of sharp changes of an unknown coefficient. The reason for this latter property is that microlocal methods enable the identification of the wave front set – a generalization of the notion of singular support of a function, which is the complement of the largest open set where a function is smooth. See [S5] for more description and a related description of geometric optics.

**Regularization**

A well posed problem has a unique solution that depends continuously on data. Inverse problems are characteristically ill-posed. To deal with this difficulty, regularization can be employed. It is an optimization method whereby the original inverse problem is changed to another well-posed problem to obtain what is mathematically characterized as the best possible approximate solution to the given inverse problem given the characteristics of the data. A typical example of a regularization method is Tikhonov regularization, see [E1].

Regularization methods are applied when: (1) there is more than one solution to the problem (typically the minimum norm solution is sought); and/or (2) the forward operator has only dense range making the inverse operator unbounded; when data is noisy or approximate it may not be
in the range of the forward operator; filtering and/or projection and/or mollifying can be applied to find an approximate solution.

For a large set of linear problems the theory yields direct methods and these methods have been extensively applied. For linear problems the regularization parameters can be chosen so that; (1) when the inverse problem has a unique solution when exact data is given; (2) when the regularization parameters approach a given value; and (3) the approximate data approaches the exact data; then the approximate inverse problems solution approaches the true solution.

Tikhonov regularization, which seeks an approximate solution while minimizing the norm of the solution can be applied in the nonlinear case. For Tikhonov regularized nonlinear problems, as well as other regularized nonlinear problems, iterative methods are often applied to get approximate solutions. These methods require a stopping criteria and nonlinearity conditions to establish convergence, see [E1], and for example [K3] and [S9].

There is a wide range of additional considerations that must be taken into account when defining the spaces that define the norms for the regularization. If the mathematical setting for the regularization is set in Hilbert space, the inverse problems solution, or image, is typically smoothed. To avoid this, for problems where discontinuities or sharp changes in the recovered parameter, or image, are expected, the non-reflexive Banach space with the BV norm may be used. This changes the methods for establishing convergence; one of the tools is Bregman distance, see, e.g. [B7] and [S9]. Bregman distance is used to establish convergence when the derivative of the cost functional is a subdifferential, which is a set rather than a single operator. Other choices for Tikhonov regularization can include sparsity constraints; this choice is used if it is expected that the exact solution, or an excellent approximate solution, can be represented by the sum of a small number of terms. A typical norm in the regularization term is $L^1$ in this case.

**Photonic Crystals and Waveguides and Inverse Optical Design**

For inverse optical design the goal is to determine a structure that has certain properties when electromagnetic waves, oscillating in the optical range and propagating through or along the surface of a dielectric material. One example is a photonic crystal where one starts with a periodic structure, that has a bandgap, that is an interval of oscillation frequencies, for which the wave decays exponentially. Then one introduces defects in the material that have the effect of enlarging the bandgap and the aim is to place the defects so as to maximize the bandgap. The goal is to have a large set of oscillating waves that have very little amplitude after passing through the structure. This problem ultimately results in maximizing the difference of two eigenvalues, a nonlinear optimization problem.

Another problem is a shape optimization problem where one seeks to design a rough surface grating coupler that will focus light to nanoscale wave lengths while maximizing the power
output of the structure, see [B9]. An overview of an adjoint method that has been utilized successfully in optimal design, together with an explanation that motivates the method steps, and some examples where the method successfully produced a useful design are in [M12].

**Statistical Methods for Uncertainty Quantification**

In inverse problems, one often encounters the following scenario. We are given noisy data, $y + \varepsilon$, where $\varepsilon$ represents the noise. The exact data $y = Kf$, where $K$ is an operator with unbounded inverse. The goal is to recover $f$. An approximation to $f$ is determined using the noisy data, by solving an optimization problem with a regularizing term, see [E1], containing a parameter, $\lambda$. The regularizing term introduces a bias in the approximate, $f_{\alpha}$, of $f$.

When the statistics of the noise, $\varepsilon$, are known, that is the mean and variance of the noise is known, under certain hypotheses, one can recover the statistics for the approximate, $f_{\alpha}$, including the statistics of the bias. This, then, is a method for quantifying the uncertainty in the approximate solution. This can be a significant aid in the interpretation of the approximate, $f_{\alpha}$. We note that all of this analysis includes the regularizing term parameter, $\lambda$, so there is reason to want to select it optimally. Often this is done by making several numerical calculations of approximate solutions. As an alternative, one can use statistical methods to make an optimal choice of the regularization parameter; one such method is cross validation, [W1], [W2]. See [T1] for more discussion about uncertainty quantification methods. See also [K4] for discussion of statistical methods in inverse problems.

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