A complex elastographic hyperbolic solver (CEHS) to recover frequency dependent complex shear moduli in viscoelastic models utilizing one or more displacement data sets

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In this paper, we present a linear marching scheme to recover frequency dependent complex shear moduli in viscoelastic models utilizing two sets of single component displacement data. The proposed method is designed to provide stable and accurate estimation of the tissue viscoelastic stiffness parameters by solving a first order complex partial differential equation. To control the exponential growth of the numerical error resulting from one of the complex coefficients in the inverse equation, a modified upwind discretization is utilized on the first order derivative terms of the target parameter. The algorithm is fully stabilized when: (1) carefully chosen multiple data sets are combined to eliminate the remaining complex coefficient that contributes to exponential error growth; and (2) a modified Tikhonov regularization is applied to the inversion method. We obtain the stability result in the $l^2$ norm so that the numerical scheme is convergent at fractional $1/2$ order. Its performance is compared with the performance of the Algebraic Inversion Model previously investigated (see [1]). We present shear modulus reconstructions from synthetic data, from laboratory phantom data and match frequency dependent complex moduli from phantom data to several viscoelastic models. Since we have previously presented phase wave speed images from interference patterns, we exhibit those images here for comparison.

Keywords: complex; shear moduli; viscoelastic; elastography

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1. Introduction

Tissue viscoelastic stiffness parameters, e.g. shear moduli and viscosities, can have a wide range of values in various pathological states. Thus the quantitative imaging of tissue biomechanical properties may provide a good indicator of cancerous or other abnormal tissues. There is extensive work carried out to image tissue stiffness parameters from movies of the interior tissue displacement created from sequences of ultrasound RF/IQ data sets or sequences of MR data sets during a variety of experiments: (1) static com-

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pression, where the tissue is slowly compressed [2–8]; (2) harmonic oscillation, where up to two harmonic sources are used to create propagating shear waves [9–18]; and (3) transient pulses, where a traveling wave is created by a time-dependent point source or line source [19–27]. See reference [28] for a review of the recent progress in this field.

In most of those elastography techniques, the medium is assumed to be purely elastic and the shear modulus is considered to be a real quantity, neglecting the effect of viscosity in shear wave propagation and shear modulus estimation. On the other hand, initial biomechanical imaging at more than one frequency excitation show that there is strong dependence of the value of the shear modulus in tissue on the vibration frequency. For example, such initial studies have been carried out for prostate [29], breast [30], liver [31] and brain elasticity [32]. The estimation of shear viscoelasticity, i.e. the complex shear modulus, can not only offer additional information about tissue stiffness changes in different pathological states but can also provide a potential indicator for malignancy and disease detection.

In this paper, we aim to develop a stable complex partial differential equation, p.d.e., solver to reconstruct the complex shear modulus from individual frequency content of the Fourier Transform of a single component of displacement data. We will test our algorithm on synthetically generated data for three viscoelastic models. Furthermore, we will utilize laboratory measured data for a gelatin phantom and fit the resulting frequency dependence to three different viscoelastic models: (1) the Kelvin-Voigt fractional derivative (KVFD) model [29, 33–35]; (2) the Spring-pot (SP) model [32, 33]; and (3) the Standard Linear Solid (SLS) model, also called the Zener model [36]. The diagrams of these three viscoelastic models are given in Figure 1.

The KVFD model and the SP model were established by the introduction of fractional calculus into the field of viscoelasticity, where a time dependence $t^{-\alpha}$ is added in the stress relaxation function. These models are not causal, that is, they do not have finite propagation speed. However, in the frequencies represented in elastography experiments, it is possible to select parameters so that a good fit to the data can be obtained. Our third model, the Standard Linear Solid (SLS) model is causal; that is, it has finite propagation speed. It features an exponential time dependence in the stress relaxation pattern.

![Figure 1. Diagrams of Viscoelastic Models: (a) Kelvin-Voigt Fractional Derivative Model; (b) Spring-pot Model; (c) Standard Linear Solid Model.](image-url)
Compared with the simple Voigt model and Maxwell model, the three viscoelastic models that we select in this paper have been shown to provide a better match over a wide range of frequencies when comparison is made with the viscoelastic response of tissue in prostate [29], brain [32] and liver [35]. Note that we will only include the viscoelastic effect for the deviatoric part of the stress-strain relationship because this is enough to provide a good data match.

Our equation systems for the three viscoelastic models in a plane strain model in 2D or in 3D are then:

**KVFD**: \[
    \rho \ddot{u} = \nabla \left( \lambda \nabla \cdot \vec{u} \right) + \nabla \cdot \left[ \mu_f \epsilon + \frac{2 \eta_f}{\Gamma(1 - \alpha_f)} \int_0^t \frac{1}{(t-s)^\alpha_f} \frac{\partial}{\partial s} \epsilon ds \right]
\]

**SP**: \[
    \rho \ddot{u} = \nabla \left( \lambda \nabla \cdot \vec{u} \right) + \nabla \cdot \left[ \frac{2 \mu_s \tau_s^{\alpha_s}}{\Gamma(1 - \alpha_s)} \int_0^t \frac{1}{(t-s)^\alpha_s} \frac{\partial}{\partial s} \epsilon ds \right]
\]

**SLS**: \[
    \rho \ddot{u} = \nabla \left( \lambda \nabla \cdot \vec{u} \right) + \nabla \cdot \left[ 2 \mu_1 \epsilon + 2 \int_0^t e^{-(t-s)/\tau} \frac{\partial}{\partial s} (\mu_2 \epsilon) ds \right],
\]

where \(\vec{u}\) is the displacement vector, \(\epsilon = \frac{1}{2} (\nabla \vec{u} + (\nabla \vec{u})^T)\) is the strain matrix, \(\mu_f\) is the shear modulus in the Kelvin-Voigt fractional derivative model, \(\mu_s\) is the shear modulus in the Spring-pot model and \(\mu_1\) and \(\mu_2\) are the shear moduli in the Standard Linear Solid Model; \(\tau_s = \eta_s/\mu_s\) is the relaxation time for the Spring-pot model and \(\tau = \eta/\mu_2\) is the relaxation time for the Maxwell element in the Standard Linear Solid model; \(\eta_f\) and \(\eta\) are viscosities. The experiments under consideration are designed to have most of the information concentrated in one component [32, 37] and we obtain data in only one image plane and for only one component, designated as \(u\); furthermore, we assume we are in 2D and make a simplification. We use the corresponding viscoelastic wave equation models for the measured component:

\[
    \rho \ddot{u} = \nabla \cdot \left[ \mu_f \nabla u + \frac{\eta_f}{\Gamma(1 - \alpha_f)} \int_0^t \frac{1}{(t-s)^\alpha_f} \frac{\partial}{\partial s} (\nabla u) ds \right] + f,
\]

\[
    \rho \ddot{u} = \nabla \cdot \left[ \frac{2 \mu_s \tau_s^{\alpha_s}}{\Gamma(1 - \alpha_s)} \int_0^t \frac{1}{(t-s)^\alpha_s} \frac{\partial}{\partial s} (\nabla u) ds \right] + f,
\]

\[
    \rho \ddot{u} = \nabla \cdot \left[ 2 \mu_1 \epsilon + 2 \int_0^t e^{-(t-s)/\tau} \frac{\partial}{\partial s} (\mu_2 \epsilon) ds \right] + f,
\]

in \(\Omega \times (0, T)\), where all the shear moduli \(\mu_f, \mu_s, \mu_1, \mu_2\), the viscosities \(\eta_f, \eta\), the relaxation times \(\tau_s, \tau\) and the density \(\rho\) are continuously differentiable. Here we have also included the force, \(f\), which represents the push used in the Acoustic Radiation force Crawling wave (ARC) experiment (see Section 4 for details). In the numerical experiments, this force is derived from the acoustic fields simulated using Field II ([38], [39]). This push is described in more detail in Section 4. For the forward problem, the medium is initially at rest:

\[
    u(x, y, 0) = u_t(x, y, 0) = 0 \quad \text{on} \quad \Omega.
\]
For the inverse problem model, since the data is movies of the displacement $u$ throughout the 2D imaging region, we assume $u(x,y,t)$ is given throughout $\Omega$ and for an interval, $0 < t \leq T$, in time. Then, we: (1) change some of the targeted parameters in the inverse problem to be the ratio between the shear moduli and density or the ratios between the viscosities and density, by making the commonly accepted assumption that $\rho(x,y)$ is a constant; and (2) extract individual frequency content of the measured data by Fourier Transforming the forward equations (1), (2) and (3) to the frequency domain:

\[
\begin{align*}
\nabla \cdot \left[ \mu_f \nabla u + \frac{\eta_f}{\Gamma(1-\alpha_f)} \int_0^t \frac{1}{(t-s)^{\alpha_f}} \frac{\partial}{\partial s} (\nabla u) ds \right] - \rho u_{tt} &= 0 \\
\nabla \cdot \left[ \frac{\mu_s \tau_s^{\alpha_s}}{\Gamma(1-\alpha_s)} \int_0^t \frac{1}{(t-s)^{\alpha_s}} \frac{\partial}{\partial s} (\nabla u) ds \right] - \rho u_{tt} &= 0 \\
\n\nabla \cdot \left[ \mu_1 \nabla u + \int_0^t e^{-(t-s)/\tau} \frac{\partial}{\partial s} (\mu_2 \nabla u) ds \right] - \rho u_{tt} &= 0 \\
\end{align*}
\]

\[
\hat{u}(x,y,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} u(x,y,t) dt
\]

\[
\left\{\begin{array}{l}
\nabla \tilde{\mu}(x,y) \cdot \nabla \hat{u}(x,y,\omega_c) + \tilde{\mu}(x,y) \triangle \hat{u}(x,y,\omega_c) + \omega_c^2 \hat{u}(x,y,\omega_c) = 0,
\end{array}\right.
\]

where

- KVFD: $\tilde{\mu} = (\mu_f + \eta_f (i\omega)^{\alpha_f}) / \rho$
- SP: $\tilde{\mu} = \mu_s (i\omega \tau_s)^{\alpha_s} / \rho$
- SLS: $\tilde{\mu} = \left( \mu_1 + \frac{\mu_2 i\omega \tau}{i\omega \tau + 1} \right) / \rho$.

It is then, a complex shear modulus that needs to be recovered as a solution to (5) in the inverse problem. In the inverse problem, notice that the force term, $f$, has been eliminated as the source term will be located outside the imaging region.

As an often employed practice in the inverse problem of elastography, a simple Algebraic Inversion model can be achieved by making the locally constant assumption on $\tilde{\mu}$ and neglecting the first derivative terms of $\tilde{\mu}$ from (5):

\[
\hat{\mu}(x,y) \triangle \hat{u}(x,y,\omega_c) + \omega_c^2 \hat{u}(x,y,\omega_c) = 0,
\]

where we denote the approximated $\tilde{\mu}$ by $\hat{\mu}$. Under some hypotheses, $\hat{\mu}$ is a good approximation to $\tilde{\mu}$; see [1] for a rigorously established bound on the relative difference. However, when there are a number of high contrast inclusions in the viscoelastic background, solving the first order partial differential equation (5) can yield a more reliable estimation of shear viscoelasticity.

This paper then considers the reconstruction of a complex shear modulus $\tilde{\mu}$ by solving the first order p.d.e.: that is, find complex valued $\tilde{\mu}$ by solving the complex differential
\[ \nabla \tilde{\mu}(x, y) \cdot \nabla \hat{u}(x, y, \omega_c) + \tilde{\mu}(x, y) \Delta \hat{u}(x, y, \omega_c) + \omega_c^2 \hat{u}(x, y, \omega_c) = 0. \] (7)

Compared with the previous development of p.d.e. solvers for recovering the real shear modulus \[\mu\] [40], the task here of recovering a complex shear modulus \(\tilde{\mu}\) from the complex first order p.d.e. (7) presents additional difficulty. The equation itself is non-hyperbolic and the exact solution of a non-hyperbolic system will exponentially grow with part of its growth rate determined by the imaginary part of the complex coefficients of the first order derivative terms of \(\tilde{\mu}\) [41, 42]. In addition, even if we assume the complex shear modulus possesses no variation in one spatial dimension and reduce the first order p.d.e. to an ordinary differential equation, the numerical error of the equation can still be exponentially growing with its growth rate determined by the real part of the complex coefficient of the zero order derivative term of \(\tilde{\mu}\), [42]. It has been shown that simple upwind discretization will fail and even a more advanced version of an upwind scheme is not stable for a non-hyperbolic system [41].

In this work, even though our equation is not hyperbolic, we propose a linear finite difference based marching scheme to reconstruct \(\tilde{\mu}\) by solving the inverse problem model as an evolution type of p.d.e. To control error growth, we use a combination of carefully selected multiple displacement data sets. The numerical instability resulting from the complex coefficients of the first order derivative terms of \(\tilde{\mu}\) is controlled by a complex implicit upwind discretization and a modified Tikhonov regularization. The remaining error growth due to the coefficient of the zero order derivative term of \(\tilde{\mu}\) can be eliminated by combining multiple data sets in a novel way. The proposed scheme, where only one first order p.d.e. is solved but the coefficients are obtained using two data sets, captures the spatial variation of the real and imaginary parts of the exact solution successfully. We present the numerical method, the reconstruction of the complex shear modulus from simulated viscoelastic wave propagation data obtained from three viscoelastic models for the cases where we have two data sets. Note that in all cases the numerical method exhibits stability but the performance can differ as we will show. We also compare the images, obtained with our algorithm, with Algebraic Inversion and show significant improvement. In addition we add noise to the data and show the effect on the images.

In addition, we apply this algorithm to data sets obtained at the Center for Biomedical Ultrasound at the University of Rochester. The experimental data is created with the new Acoustic Radiation Crawling (ARC) wave experiment performed on a gel phantom with an inclusion. For the laboratory data we utilize the frequency content from 8 frequencies and provide parameter choices determined by an \(L^2\) optimization best fit for each of the viscoelastic models. Finally, for the laboratory data, we compare the complex modulus recoveries with the interference pattern phase wave speed recoveries.

The structure of this paper is as follows: In Section 2 we present finite difference discretization of the numerical scheme and the modified Tikhonov regularization. We exhibit better results when we use multiple data sets. In Section 3 we describe the viscoelastic wave equation simulation and give the numerical reconstruction of the complex shear modulus from synthetic data including results when we add noise to the synthetic data. In Section 4 the experiment is described. Section 5 is devoted to the complex shear modulus recovery from phantom data and the matching of frequency dependence to our three viscoelastic models. In Section 6, we give a comparison of the complex modulus recovery with the phase wave speed recovery. We do this since we have previously presented phase wave speed recoveries from interference patterns. Finally, we conclude this document and give ideas on future work.
2. Linear Explicit-Implicit Upwind Scheme

The first order complex equation we use for the recovery of the complex shear modulus \( \tilde{\mu} \) for individual frequency content can be changed to an evolution type of p.d.e. in the following way assuming the \( x \) direction is a pseudo-time direction and \( \hat{u}_x \neq 0 \):

\[
\hat{\mu}_x \hat{u}_x + \hat{\mu}_y \hat{u}_y + \hat{\mu} \Delta \hat{u} = -\omega^2 \hat{u} \implies \tilde{\mu}_x + a\tilde{\mu}_y + b\tilde{\mu} = c,
\]

where \( a = \hat{u}_y(\hat{u}_x)^{-1} \), \( b = \Delta \hat{u}(\hat{u}_x)^{-1} \), \( c = -\omega^2 \hat{u}(\hat{u}_x)^{-1} \) are all complex functions. We can discretize the equation above with the following finite difference scheme where we assume uniform discretization in \( y \) and denote the solution of the discretized problem on the grid as \( \tilde{\mu}_{i,j} \), \( 1 \leq i \leq M, 1 \leq j \leq N \):

\[
\frac{\tilde{\mu}_{i+1,j} - \tilde{\mu}_{i,j}}{dx_i} + a_{i,j} \begin{cases} \frac{\tilde{\mu}_{i+1,j+1} - \tilde{\mu}_{i+1,j}}{dy} & \text{Re}(a_{i,j}) < 0 \\ \frac{\tilde{\mu}_{i+1,j} - \tilde{\mu}_{i,j-1}}{dy} & \text{Re}(a_{i,j}) \geq 0 \end{cases} + b_{i,j} \tilde{\mu}_{i,j} = c_{i,j}.
\]

where \( dx_i \) is the variable pseudo time step size. This explicit-implicit scheme can be rewritten as:

\[
A_{i,k} \tilde{\mu}_{i+1} = c_i dx_i + B_i \tilde{\mu}_i,
\]

where \( A_i = (A_{i,k,l}) \), \( 1 \leq k \leq n, 1 \leq l \leq n \) is a tridiagonal matrix with its nonzero entries defined by the following formulas:

\[
A_{i,k,k} = 1 + \text{sign}(\text{Re}(a_{i,k})) a_{i,k} \frac{dx_i}{dy},
\]

\[
A_{i,k,k+1} = -\frac{\text{sign}(\text{Re}(a_{i,k})) + 1}{2} a_{i,k} \frac{dx_i}{dy},
\]

\[
A_{i,k,k-1} = -\frac{\text{sign}(\text{Re}(a_{i,k})) - 1}{2} a_{i,k} \frac{dx_i}{dy},
\]

and \( B_i \) is a diagonal matrix whose entries are defined by \( (B_i)_{k,k} = 1 - b_{i,k} dx_i \). Notice that \(|A_{i,k,k}| \geq 1 \) and for every pair of \( A_{i,k,k+1} \) and \( A_{i,k,k-1} \), there is only one of them that is nonzero. The stability and accuracy of the matrix inversion is determined by the norm of the inverse of \( A_i \), i.e., the eigenvalues of \( A_i^T A_i \). Now let \( (\xi_1^i, \eta_1^i) \) be the eigenvalue and eigenvector pairs of \( A_i^T A_i \) and let \( 0 < \xi_1^i \leq \xi_2^i \leq \cdots \leq \xi_n^i \). If \( \xi_1^i \geq 1 \), then we solve (9) by direct matrix inversion. Now suppose \( 0 < \xi_1^i \leq 1 \), we solve the following equation instead:

\[
A_i \tilde{\mu}_{i+1} \equiv \sqrt{\xi_1^i} \left( c_i dx_i + B_i \tilde{\mu}_i \right) = \min(1, \sqrt{\xi_1^i}) \left( c_i dx_i + B_i \tilde{\mu}_i \right).
\]

This is equivalent to Tikhonov regularization on the matrix inversion, i.e., finding the optimal diagonal matrix \( D_i \) s.t.

\[
\| (D + A_i^T A_i)^{-1} A_i^T \|_2 \leq 1
\]
To stabilize the matrix inversion, we consider Tikhonov regularization in $l_2$ norm.

**Lemma 2.1** There exists a diagonal matrix $D_i$ for each $A_i$, s.t.

$$\|\tilde{\mu}_i\|_2 \leq \|\tilde{g}_i\|_2$$

for $(D + \bar{A}_i^T A_i)\tilde{\mu}_i = \bar{A}_i^T \tilde{g}_i$.

**Proof.** Consider the matrix inversion as a $L^2$ minimization problem. That is, to find $\tilde{\mu}_i$ such that

$$\min_{\tilde{\mu}_i \in \mathbb{C}^n} \|A_i \tilde{\mu}_i + 1 - \bar{c}_i dx_i - B_i \tilde{\mu}_i\|_2^2 = \min_{\tilde{\mu}_i \in \mathbb{C}^n} I(\tilde{\mu}_i).$$

Assume $\tilde{\mu}_i$ as known and let $\tilde{g}_i = \bar{c}_i dx_i + B_i \tilde{\mu}_i$. Then the minimization problem $\min_{\tilde{\mu}_i \in \mathbb{C}^n} I(\tilde{\mu}_i)$ becomes

$$\min_{\tilde{\mu}_i \in \mathbb{C}^n} \|A_i \tilde{\mu}_i + 1 - \tilde{g}_i\|_2^2.$$

To minimize, we need to solve for

$$(A_i^T A_i)\tilde{\mu}_{i+1} + 1 = \bar{A}_i^T \tilde{g}_i.$$  \hfill (12)

Now let $(\xi^i_j, \bar{\eta}^i_j)$ be the eigenvalue and eigenvector pair of $\bar{A}_i^T A_i$ and let $0 < \xi^i_1 \leq \xi^i_2 \leq \cdots \leq \xi^i_n$. If $\xi^i_1 \geq 1$, then we can solve for

$$\tilde{\mu}_{i+1} = A_i^{-1} \tilde{g}_i \quad \text{and} \quad \|\tilde{\mu}_{i+1}\|_2 \leq \|\tilde{g}_i\|_2.$$

Now suppose $0 < \xi^i_1 \leq 1$, we can change the minimization problem by representing $\tilde{\mu}_{i+1} = \sum_{j=1}^{n} \bar{\eta}^i_j(\tilde{\mu}_{i+1}, \bar{\eta}^i_j)$ and get

$$\min_{\tilde{\mu}_i \in \mathbb{C}^n} \left[ \|D \tilde{\mu}_i\|_2^2 + \|A_i \tilde{\mu}_{i+1} - \tilde{g}_i\|_2^2 \right] = \min_{\tilde{\mu}_i \in \mathbb{C}^n} \left[ \| \sum_{i=1}^{n} \sqrt{d_j(\tilde{\mu}_{i+1}, \bar{\eta}^i_j)} \bar{\eta}^i_j\|_2^2 + \|A_i \tilde{\mu}_{i+1} - \tilde{g}_i\|_2^2 \right]$$

$$= \min_{\tilde{\mu}_i \in \mathbb{C}^n} I_{\alpha}(\tilde{\mu}_i), \text{ where } \alpha > 0,$$

where $D$ is a diagonal matrix with diagonal element $d_j > 0$. In this case, we have

$$(D + \bar{A}_i^T A_i)\tilde{\mu}_i = \bar{A}_i^T \tilde{g}_i,$$  \hfill (13)

which is equivalent to

$$\tilde{\mu}_{i+1} = \sum_{j=1}^{n} \left( \frac{\xi^i_j}{d_j + \xi^i_j} \right) (A_i^{-1} \tilde{g}_i, \bar{\eta}^i_j) \bar{\eta}^i_j \Rightarrow$$
\[ \|\tilde{\mu}_i\|_2^2 = \sum_{j=1}^{n} \left( \frac{\xi_j^i}{d_j + \xi_j^i} \right)^2 \left| (A^{-1}\tilde{g}_i, \tilde{\eta}_j) \right|^2 \]

\[ \leq \left[ \max_j \left( \frac{\xi_j^i}{d_j + \xi_j^i} \right) \right] \|A^{-1}\tilde{g}_i\|_2^2. \]

Since \( \|A^{-1}\|_2 = \frac{1}{\sqrt{\xi_1}} \), if we choose \( d_j = \xi_j^i \left[ (\xi_1^i)^{-1/2} - 1 \right] \), we will have

\[ \frac{\xi_j^i}{(d_j + \xi_j^i)\sqrt{\xi_1^i}} = 1 \Rightarrow \|\tilde{\mu}_i\|_2 \leq \|\tilde{g}_i\|_2. \]

\[ \square \]

The structure of the matrix \( A^i \) ensures row diagonal dominance:

\[ |A_{k,k}^i| = \sqrt{\left( 1 + |\text{Re}(a_{i,k})| \frac{dx_i}{dy} \right)^2 + \left( |\text{Im}(a_{i,k})| \frac{dx_i}{dy} \right)^2} \]

\[ > \sum_{l \neq k} |A_{k,l}^i| = \sqrt{\left( |\text{Re}(a_{i,k})| \frac{dx_i}{dy} \right)^2 + \left( |\text{Im}(a_{i,k})| \frac{dx_i}{dy} \right)^2} = |a_{i,k}| \frac{dx_i}{dy} \]

3. Numerical Analysis

3.1. Stability

We first present the stability analysis of the linear explicit-implicit hyperbolic scheme and then the fractional order convergence result will follow with the regularization term. Consider the homogeneous part of (8). We assume \( \tilde{\mu} \in C^2(R) \) satisfies:

\[ \tilde{\mu}_x + a(x,y)\tilde{\mu}_y + b(x,y)\tilde{\mu} = 0, \]

where the domain \( R = \{(x,y)|0 \leq x \leq L, 0 \leq y \leq L^*\} \) and \( a, b \in C(R) \). Let \( \tilde{\mu}_i \) be the numerical solution at the \( i \)th step \( x_i \). We first seek stability results where we show there exists \( C, \hat{C} > 0 \) with \( \|\tilde{\mu}_{i+1}\|_2 \leq C\|\tilde{\mu}_i\|_2 \) so that independent of step size, at the final \( n \)th step \( \|\tilde{\mu}_n\|_2 \leq \hat{C}\|\tilde{\mu}_0\|_2 \). Nevertheless, the constant \( C \) can be large. We improve on the constant later in the paper by utilizing optimal data sets.

Lemma 3.1  There exists \( \alpha \) independent of step sizes such that

\[ \|\tilde{\mu}_{i+1}\|_2 \leq e^{\alpha dx_i} \|\tilde{\mu}_i\|_2, \]

where \( \alpha = \max_{i,j} |b_{i,j}| \). Further more \( \|\tilde{\mu}_n\|_2 \leq e^{\alpha L} \|\tilde{\mu}_0\|_2. \)
Proof. If equation (14) is solved by regularized numerical scheme (11), we obtain:

$$\|\tilde{\mu}_{i+1}\|_2 \leq \sqrt{\xi_i} |A_i^{-1}|_2 \cdot \max_j |1 - b_{i,j} dx_i| \cdot \|\tilde{\zeta}_i\|_2 \leq e^{\alpha dx_i} \|\tilde{\mu}_i\|_2.$$  

The second inequality follows directly. 

This proves the stability of the linear numerical scheme in the $l^2$ norm (see (2.4.2) from [43]).

3.2. Error Control with Combination of Multiple Data Sets

To completely control the growth of the numerical solution, we invoke the concept of combining multiple displacement data sets to eliminate the zero order derivative term of $\tilde{\mu}$ from the first order p.d.e. that governs the inverse problem solution.

Suppose we have two individual sets of displacement data, $u_j$, $j = 1, 2$, obtained from two separate experiments for the same medium, then both data sets satisfy the same first order equation in the inverse problem:

$$\begin{cases} 
\hat{u}_{j,x} \tilde{\mu}_x + \hat{u}_{j,y} \tilde{\mu}_y + \mu \Delta \hat{u}_j + \omega^2 \hat{u}_j = 0, \ j = 1, 2. 
\end{cases}$$

By the following simple algebra step, we obtain:

$$\begin{align*}
\Delta \hat{u}_2 (\hat{u}_{1,x} \tilde{\mu}_x + \hat{u}_{1,y} \tilde{\mu}_y + \mu \Delta \hat{u}_1 + \omega^2 \hat{u}_1) &= 0 \\
-\Delta \hat{u}_1 (\hat{u}_{2,x} \tilde{\mu}_x + \hat{u}_{2,y} \tilde{\mu}_y + \mu \Delta \hat{u}_2 + \omega^2 \hat{u}_2) &= 0 \\
(\Delta \hat{u}_2 \hat{u}_{1,x} - \Delta \hat{u}_1 \hat{u}_{2,x}) \tilde{\mu}_x + (\Delta \hat{u}_2 \hat{u}_{1,y} - \Delta \hat{u}_1 \hat{u}_{2,y}) \tilde{\mu}_y &= -\omega^2 (\Delta \hat{u}_2 \hat{u}_1 - \Delta \hat{u}_1 \hat{u}_2)
\end{align*}$$

Equation (15) is a simple transport equation with no zero order derivative term of $\tilde{\mu}$ and if we use the same complex upwind discretization on the equation above, we will get:

$$A^T \tilde{\mu}_{i+1} = \tilde{c}_i dx_i + \tilde{\mu}_i.$$  

In this case, the contribution of solution growth from the $b_{i,j}$ term is gone as the zero order derivative term of $\tilde{\mu}$ is eliminated from the equation governing the solution of the inverse problem. Provided that $(\Delta \hat{u}_2 \hat{u}_{1,x} - \Delta \hat{u}_1 \hat{u}_{2,x})$ is bounded away from zero, the numerical scheme is stable and the constant $e^{\alpha L}$ in Lemma 3.1 is significantly reduced. Examples will be given in the next section to exhibit the superior stability control of this data combination.
4. Performance Test on Simulated Data

In this section, we test the linear explicit-implicit hyperbolic scheme and compare its performance depending on which marching direction is chosen and whether or not we have one or two data sets. We also compare with the result obtained using the Algebraic Inversion method.

The forward simulation has many similar features as that given in [37]: here we solve the viscoelastic wave equations for all three viscoelastic models

\[
\text{KVFD} : \quad \rho u_{tt} = \nabla \cdot \left[ \mu_f \nabla u + \frac{\eta_f}{\Gamma(1 - \alpha_f)} \int_0^t \frac{1}{(t-s)^\alpha_f} \partial_s (\nabla u) ds \right] + f, \\
\text{SP} : \quad \rho u_{tt} = \nabla \cdot \left[ \frac{\mu_s \tau_s^{\alpha_s}}{\Gamma(1 - \alpha_s)} \int_0^t \frac{1}{(t-s)^\alpha_s} \partial_s (\nabla u) ds \right] + f, \\
\text{SLS} : \quad \rho u_{tt} = \nabla \cdot \left[ \mu_1 \nabla u + \int_0^t e^{-{(t-s)/\tau}} \partial_s (\mu_2 \nabla u) ds \right] + f.
\]

In this study for our forward simulations, we allow all stiffness parameters (with the exception of \(\alpha_f, \alpha_s,\) and \(\tau\)) in the viscoelastic model to be spatially varying, which effectively provides spatial dependence for both the real and imaginary parts of the complex shear modulus \(\tilde{\mu}\). The forcing function \(f\) for all three models is generated by Field II, a program for simulating ultrasound transducer fields and ultrasound imaging using linear acoustics. To simulate the acoustic radiation force used in the ARC experiment, we used the physical parameters of a specially built transrectal transducer as inputs to the Field II simulation. The pitch of the transducer array was 203 microns and the center frequency was approximately 5 MHz. Using these parameters, a two-dimensional intensity function was calculated using Field II. The simulated intensity map is assumed to be proportional to the applied force, using the following equation ([44]):

\[
f = \frac{2\tilde{\alpha} I}{c}
\]

where \(\tilde{\alpha}\) is the absorption coefficient, \(I\) is the temporal average intensity, and \(c\) is the speed of sound in water. Using this relationship, the simulated intensities can be used to derive the force for a 250 microsecond pushing pulse as used in the experiment (see Section 5 for more details about the experiment). We note that with this forcing function we do not compute the scattered field as in [37] and then add the scattered field and a closed form incident field. Here we compute the total displacement directly.

We use a second order finite difference scheme in time and space to discretize the displacement in the forward equation. We assume the Sommerfeld radiation condition for the wave and implement a second-order split-field perfectly matched layer absorbing boundary condition around the computational domain to prevent numerical reflections of outgoing waves (see [37, 45]). To avoid numerical problems in the inverse problem, the so called “inverse crime”, we use the synthetic data from the forward simulation on the forward problem computational grid to generate a new set of displacement data on a different grid. To do this for each time frame, we first make a two-dimensional cubic spline interpolation and then sample the interpolated displacement on a new mesh. The grid size of the new mesh is two thirds of the grid size of the old mesh used in the forward
simulation.

We Fourier Transform the data set in time and extract content at each of several frequencies. In each case, to compute the numerical derivatives of the complex data, we first separate the phase and the amplitude as \( \hat{u} = Me^{i\phi} \). The derivatives of \( \hat{u} \) are then calculated in terms of derivatives of \( M \) and \( \phi \) by an averaging method (see [46]) with a 3 by 3 window to eliminate numerical noise. To resolve the jump discontinuities in the wrapped phase \( \phi \), we utilize an \( L^1 \) minimization procedure (see [47]) to perform multidimensional phase unwrapping.

For the linear hyperbolic forward problem p.d.e. solver, we generate two sets of data from two different simulations: one with the Field II push on top of the domain and the other with the Field II generated push at the bottom of the domain. The time step size used during the simulation is 0.5ms. These two individual data sets and their numerical derivatives are used separately or together, as described in Section 3, and then fed to the linear p.d.e. solver. For the Algebraic Inversion method, we only use one of those two data sets to compute numerical derivatives and the application of it is much more straightforward. The results given in this section are obtained from frequency contents at 20.80Hz for the KVFD model, 16.65Hz for the SP model and 19.13Hz for the SLS model.

For all three viscoelastic models, the real and imaginary parts of the exact shear modulus have two elliptical inclusions in a constant background. The input parameters to the simulation are defined using the following procedure:

\[
g(x, y) = A \max \left( 0.0015^2 - \frac{(x \cos(2\pi/5) + (y + 0.0025) \sin(2\pi/5))^2}{0.0025}, 0 \right) +
B \max \left( 0.003^2 - \frac{(x \sin(2\pi/5) - (y + 0.0025) \cos(2\pi/5))^2}{0.02}, 0 \right) +
\]

\[
\mu_f = g(x, y), \mu_0 = 1, \eta_f = 0.5\mu_f, \alpha_f = 0.25, A = 0.25e + 6, B = 0.6e + 5
\]

\[
\mu_s = g(x, y), \mu_0 = 2, \tau_s = 0.002, \alpha_s = 0.25, A = 1.2e + 6, B = 0.3e + 6
\]

\[
\mu_1 = 1, \mu_2 = g(x, y), \mu_0 = 2, \tau = 0.01, A = 1.2e + 6, B = 0.3e + 6
\]

We present numerical reconstruction results from three synthetic data sets, two data sets for each of the three viscoelastic models. In Figure 2, we show the true values of the real and imaginary parts of the modulus \( \hat{\mu} \) in the KVFD, SP and SLS models.

In Figure 3, we present reconstruction results that are computed from two data sets for each model using equation (15). Note that the recoveries of the real and imaginary parts of the complex modulus captures the size and the location of the inclusion perfectly while undershoots the amplitude of the exact value. This is due to the regularization effect on the growth of the numerical solution. Lastly, the Algebraic Inversion results are
Figure 2. Exact value of complex modulus at frequency 25Hz (KVFD), 20Hz (SP) and 12.5Hz (SLS): (a) real part, KVFD model; (b) real part, SP model; (c) real part, SLS model; (d) imaginary part, KVFD model; (e) imaginary part, SP model; (f) imaginary part, SLS model. Notice that in all cases, the imaginary part is smaller than the real part. The units for the images are KPa.

presented in Figure 4. It is very clear that the Algebraic Inversion is much less reliable in recovering the complex shear modulus in this practical simulation set up.

Figure 3. Recovery from two data sets: (a) real part, KVFD Model; (b) imaginary part, KVFD Model; (c) absolute value of the error of real part, KVFD Model; (d) absolute value of the error of imaginary part, KVFD Model; (e) real part, SP Model; (f) imaginary part, SP Model; (g) absolute value of the error of real part, SP Model; (h) absolute value of the error of imaginary part, SP Model; (i) real part, SLS Model; (j) imaginary part, SLS Model; (k) absolute value of the error of real part, SLS Model; (l) absolute value of the error of imaginary part, SLS Model. The units for the images are KPa. Note that the color bars in the error plots have different ranges.
Figure 4. Recovery from Algebraic Inversion: (a) real part, KVFD Model; (b) imaginary part, KVFD Model; (c) absolute value of the error of real part, KVFD Model; (d) absolute value of the error of imaginary part, KVFD Model; (e) real part, SP Model; (f) imaginary part, SP Model; (g) absolute value of the error of real part, SP Model; (h) absolute value of the error of imaginary part, SP Model; (i) real part, SLS Model; (j) imaginary part, SLS Model; (k) absolute value of the error of real part, SLS Model; (l) absolute value of the error of imaginary part, SLS Model. The units for the images are KPa.

Finally we present results from noisy synthetic data. We added 1% Gaussian noise to the synthetic data and reconstructed the complex modulus using our regularized complex elastographic hyperbolic solver. In Figure 5 the recovery from two data sets are demonstrated. Our numerical solver captures the location and the size of the inclusions while the amplitude is underestimated. Note that the imaginary part of the shear modulus degrades, with the addition of noise, more than the real part of the shear modulus.

5. Phantom Experiments

The Acoustic Radiation force Crawling wave (ARC) experimental investigation was conducted at the University of Rochester. A GE Logiq 9 ultrasound system was modified to collect the data required to generate the synthetic ARC wave displacement time histories. A special research scan sequence format was developed to allow a sequence of pushing and tracking vectors to be fired with the desired timing. The duty cycle of the overall scan sequence was 0.35% to avoid thermal limits of the components. The sequence is designed to provide displacement data for a region of interest (ROI) that is typically about 18 mm, though this can be adjusted. The ROI is made up of 31 vectors or locations for which the data was collected. Figure 6 shows the layout of the ROI. The 31 vectors are spaced equally across the ROI, with a spacing of roughly 0.6 mm. For each vector location there is a series of push and track firings. Figure 7 shows the scan sequence that occurs for each location. This sequences consists of a set of reference firings, a push vector on the left side of the ROI, followed by a series of tracking vectors. This is followed by a pause to maintain the duty cycle below thermal limits. Then a push pulse is fired on the right side of the ROI, and this push is also followed by a series of tracking vectors. These
tracking vectors are followed by another pause to maintain the duty cycle. This sequence is repeated for each location in the ROI. The left and right push locations are the same for all the vectors in the ROI, but the tracking locations are moved across the ROI. The tracking vectors are fired at a pulse repetition rate of 2.5 kHz. The entire sequence for all 31 locations takes about 4.5 seconds to collect. Complex baseband demodulated data (IQ) for each of the reference and tracking vectors was stored for offline processing. The sampling rate of the IQ data is 10 MHz. The IQ data was processed to calculate the axial (toward or away from the transducer) component of the displacement for each location as a function of time.

A gelatin phantom with homogeneous background (approximately 5% gelatin) and a finger inclusion of higher stiffness (approximately 10% gelatin) was constructed. The gelatin phantom with the finger inclusion was scanned using the specially designed scan sequences on the modified GE Logiq 9 using a specially built transrectal probe. Two sets of displacement data where collected in rapid succession using the sequence described above. One generated a pushing force which was peaked on the left side of the region of interest (ROI) and the other generated a pushing force which peaked on the right side of the region of interest. The entire sequence was repeated multiple times and averaged to improve the signal to noise ratio. The end result is two axial displacement versus time profiles for the 2D ROI, \( u(x, y, t) \) and \( v(x, y, t) \).

6. Performance Test on Phantom Data

The time trace from each displacement data set contains 48 time steps with 0.3ms spacing between samples. We synthetically extend the time trace to 400 time steps setting the displacement equal to zero where we have no data measurements. Then we Fourier Transform the data in time. We use the extended and transformed data from both displacements at each of nine frequencies to reconstruct the complex modulus with our
Figure 6. Region of Interest (ROI) in the experimental set up. The depth is 40mm and width is 18mm. The focal depth is 25mm. The ROI is made up of 31 vectors or locations where data is collected.

Figure 7. Graphic display of scan sequence at each location. Horizontal axis is time.

complex marching p.d.e. solver for equation (8). Unlike the case of synthetic data, we applied scheme (12) instead of (13). We removed the regularization here since the additional regularization on the numerical solution often gives unacceptably over-regularized results from the exceptionally noisy measured phantom data. The acceptance criteria of numerical results comes from the comparison between our numerical recoveries and the targeted viscoelastic parameters used in [48]. After we obtained the real and the imaginary parts of the complex modulus, we use a total variation minimization procedure (see [49]) to improve the contrast between the high speed and background regions.

In Figure 8, we present the recovery of the real and imaginary parts of complex shear
modulus for this finger inclusion phantom from the complex marching p.d.e. solver and the Algebraic Inversion method together with the B-mode image. For each image, the vertical axis extends from 0.0192m to 0.0338m with a discretization of 0.000077m while the horizontal axis extends from 0.003m to 0.015m with a discretization of 0.006m. The lower bound of the colorbar is 15m/s and the upper bound of the colorbar is 40m/s. The frequency content utilized in this recovery is obtained at 243.75Hz and 268.75Hz. Compared with the Algebraic Inversion result, the linear hyperbolic solver with two data sets has a recovery that is more consistent with the size and location of the inclusion. The location of the recovery is however somewhat shifted. Notice that the real part of reconstruction results obtained with the complex marching p.d.e. solver and two data sets in these two different frequencies exhibit a better correlation with the B-mode image than the imaginary part of the recoveries. This is consistent with the results from synthetic data as the imaginary part of the the complex modulus is generally more difficult to reproduce than the real part.

![B-mode Image](image1)

![Recovery from Phantom Data](image2)

Figure 8. Recovery from Phantom Data: (a) B-mode image; (b) real part, p.d.e. solver at 243.75Hz; (c) real part, p.d.e. solver at 268.75Hz; (d) real part, Algebraic Inversion at 268.75Hz; (e) imaginary part, p.d.e. solver at 243.75Hz; (f) imaginary part, p.d.e. solver at 268.75Hz; (g) imaginary part, Algebraic Inversion at 268.75Hz. The units for the shear modulus images are KPa.

With the recovery of complex shear modulus $\tilde{\mu}$ at 9 different frequencies, we calculate the average value of the real and imaginary parts of the shear modulus inside the inclusion where the location of the inclusion is determined by edge detection in the B-mode image. Then we determine the viscoelastic parameters in all three viscoelastic models by minimizing the error with a least-square fit:

$$\chi = \frac{1}{N} \sum_{i=1}^{N} \sqrt{[\text{Re}(\tilde{\mu}(\omega_i) - \tilde{\mu}_M(\omega_i))]^2 + [\text{Im}(\tilde{\mu}(\omega_i) - \tilde{\mu}_M(\omega_i))]^2},$$
where $\tilde{\mu}_M(\omega_i)$ is the exact model-based complex shear modulus that is determined by:

\[
\begin{align*}
\text{KVFD} : & \quad \tilde{\mu}_M(\omega_i) = \mu_f + \eta_f (i\omega_i)^{\alpha_f} \\
\text{Spring-pot} : & \quad \tilde{\mu}_M(\omega_i) = \mu_s (i\omega_i \tau_s)^{\alpha_s} \\
\text{Zener} : & \quad \tilde{\mu}_M(\omega_i) = \mu_1 + \frac{\mu_2 i\omega_i \tau_s}{i\omega_i \tau_s + 1}
\end{align*}
\]

with $\mu_f, \eta_f, \alpha_f, \mu_s, \tau_s, \alpha_s, \mu_1, \mu_2$ and $\tau$ being the undetermined parameters. In Figure 9, the average value of the real and imaginary parts of the complex shear modulus recovery inside the inclusion is plotted against frequency together with the fitted curves for all three viscoelastic models. The plots of the averaged values exhibit an increasing trend with the increase of frequency. We give our fitting results of the viscoelastic parameters in Table 1. We note that the best-fit curves capture the increasing trend of the shear modulus values. In Figure 10, for completeness, we show the wave speed plot calculated from the average complex modulus values using a local plane wave assumption[36], along with the wave speed plots calculated from the best-fit complex modulus curves using a local plane wave assumption:

\[
c(\omega) = \frac{1}{Re\sqrt{\frac{1}{\tilde{\mu}(\omega)}}}.
\]

In Figure 9, the average value of the real and imaginary parts of the complex shear modulus recovery inside the inclusion is plotted against frequency together with the fitted curves for all three viscoelastic models. The plots of the averaged values exhibit an increasing trend with the increase of frequency. We give our fitting results of the viscoelastic parameters in Table 1. We note that the best-fit curves capture the increasing trend of the shear modulus values.

Table 1. Fitted Viscoelastic Parameters

<table>
<thead>
<tr>
<th>Model</th>
<th>$\mu_f$</th>
<th>$\eta_f$</th>
<th>$\alpha_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>10.03KPa</td>
<td>0.003KPa·s</td>
<td>0.84</td>
</tr>
<tr>
<td>FD Model</td>
<td>$\mu_s$</td>
<td>$\tau_s$</td>
<td>$\alpha_s$</td>
</tr>
<tr>
<td>Value</td>
<td>5.18KPa</td>
<td>0.01s</td>
<td>0.50</td>
</tr>
<tr>
<td>SLS Model</td>
<td>$\mu_1$</td>
<td>$\mu_2$</td>
<td>$\tau$</td>
</tr>
<tr>
<td>Value</td>
<td>11.07KPa</td>
<td>77.13KPa</td>
<td>0.0001s</td>
</tr>
</tbody>
</table>

7. Comparison with Phase Wave Speed

In the previous sections, we used the two data sets to recover the complex shear modulus. In this section, we compare those results to the results obtained when we combine the two data sets in a different way and recover the phase wave speed.

The two data sets can be combined to create a synthetic interference pattern. The idea here is to obtain something similar to a crawling wave using two independently collected data sets. We apply the delay/add procedure described above to synthetically extend
4.2 Each data set in time, using a time shift of $N\Delta t$, obtaining

$$u_N(x, y, t) = \sum_n u(x, y, t - nN\Delta t), \quad v_N(x, y, t) = \sum_n v(x, y, t - nN\Delta t).$$

After filtering around the repetition frequency $\omega_N = 2\pi/(N\Delta t)$, we have

$$\tilde{u}_N(x, y, t) \approx a_N(x, y) \cos[\omega_N(t + \phi_N(x, y))], \quad \tilde{v}_N(x, y, t) \approx b_N(x, y) \cos[\omega_N(t - \phi_N(x, y))].$$

The synthetic interference pattern is computed as the variance over one period of the sum of the left time trace and a time-delayed version of the right time trace:

$$w_N(x, y, t) = \frac{1}{N\Delta t} \int_{t_0}^{t_0 + N\Delta t} [\tilde{u}_N(x, y, s) + \tilde{v}_N(x, y, t + s)]^2 ds.$$
After filtering to remove the zero-frequency terms, we have

\[ \tilde{w}_N(x, y, t) \approx a_N(x, y)b_N(x, y)\cos[\omega_N(-t + \phi_N(x, y) + \varphi_N(x, y))]. \]

Thus the phase wave speed of the interference pattern is

\[ c_N(x, y) = \frac{1}{|\nabla \phi_N(x, y) + \nabla \varphi_N(x, y)|} \]

where \( c_s \) is the shear wave speed and \( \theta \) is the angle between \( \nabla \phi_N \) and \( \nabla \varphi_N \). This relation is similar to the relation between the crawling wave speed and shear wave speed in [47].

We recover the interference pattern phase wave speed using either an \( L^1 \)-minimization phase-unwrapping method or a local cross-correlation method. Both of these methods are described in [47]. We approximate the angle \( \theta \) by 0, and then we obtain an approximation of the shear wave speed using

\[ c_s \approx 2c_N. \]

In this case, \( \theta \approx 0 \) is a reasonable assumption because the two waves are nearly plane waves propagating in opposite directions in the imaging region. If this were not the case, we would recover the shear wave speed from the interference pattern using the method described in [37].

In Figure 11, we show the phase wave speed images that we obtain from the interference pattern using time shifts \( N\Delta t = 12\Delta t \) and \( N\Delta t = 8\Delta t \). These time shifts correspond to repetition frequencies of \( \omega_N = 2\pi \times 277.7 \) and \( \omega_N = 2\pi \times 416.6 \), respectively. Since the images obtained from the \( L^1 \)-minimization phase unwrapping method and the local cross-correlation method are nearly identical, we only show one set of images. These images illustrate that we are able to identify the stiff region using the phase wave speed method, and that we are able to capture the frequency dependence of the phase wave speed. We note that the inclusion can appear elongated in the vertical direction. This happens occasionally for the complex modulus recovery method presented in the previous sections of this paper. However, this elongation happens much more frequently in the phase wave speed recoveries. The source of this elongation is still under investigation.

![Figure 11. B-mode image (left) and recovered shear wave speed images using repetition frequencies 277.7 Hz (left) and 416.6 Hz (right), corresponding to time shifts of \( N = 12 \) time steps and \( N = 8 \) time steps, respectively. The units on the horizontal and vertical axes are meters and the units on the colorbar are meters/second.](image-url)
8. Conclusion and Future Work

We developed a linear explicit-implicit hyperbolic solver for recovering the complex shear modulus from a first order complex p.d.e. model. We show the numerical scheme is stable with a combination of multiple displacement data sets and a modified Tikhonov regularization term. Compared with the Algebraic Inversion method previously investigated (see [1]), the linear hyperbolic p.d.e. solver is more stable when the shear modulus is rapidly changing. To preserve the sharp contrast of the shear modulus between the phantom inclusion and the gel background, we applied the complex p.d.e. solver to the phantom data sets without regularization. The growth of the numerical solution is under control and better reconstructions of the shape and size of the inhomogeneity have been yielded than the Algebraic Inversion method. Compared with the phase wave speed recovered from the synthetic interference pattern, the complex modulus recoveries show less elongation of the inclusion in the vertical direction. The average value of the real and imaginary parts of the complex shear modulus recoveries exhibits frequency dependence consistent with viscoelastic models. We have also provided viscoelastic parameter choices for three specific viscoelastic models by a least-square fitting of the complex shear modulus recovery. For the gel phantom experiments, all three models show a similar qualitative trend for the frequency dependent complex shear modulus.

8.1. Acknowledgements

The project was partially supported by NIA Grant No. R01AG029804. J. McLaughlin and A. Thomas were partially supported by Grant No N000 14-08-1-0432 from ONR. The contents of this paper are solely the responsibility of the authors and do not necessarily represent the official views of ONR and NIA. A. Thomas was also partially supported by a NPSC Fellowship. J. McLaughlin and A. Thomas benefited from their participation during Fall, 2010, semester special program on inverse problems at MSRI. The authors would like to thank Ning Zhang, Ben Szajewski, and Yixiao Zhang, who are former students at Rensselaer and Dr. Antoinette M. Maniatty and Dr. Assad A. Oberai in MANE of RPI for useful discussions.

References

References

[7] O'Donnell M, Skovoroda AR, Shapo BM, and Emelianov SY. Internal displacement and


9. Additional features

9.1. Title, authors’ names, abstract and keywords

The title should be generated at the beginning of your article using the \maketitle command. In the final version, the author name(s) and affiliation(s) must be followed immediately by \maketitle as shown below in order for them to be displayed in your PDF document. To prepare an anonymous version for peer review, you can put the \maketitle between the \title and the \author in order to conceal the authors’ identities from the reviewers. Next you should include an abstract, which should be enclosed within an abstract environment. For example, the titles for this guide begin as follows:

\begin{abstract}
This guide is for authors who are preparing papers for \textit{Inverse Problems in Science and Engineering} (\textit{gIPE}) using the \LaTeX\ document preparation system and the class file \texttt{gIPE2e.cls}, which is available via the journal’s home page on the Taylor \& Francis website. \end{abstract}

A list of keywords should follow the abstract, enclosed within a keywords environment, followed by subject classification codes enclosed within a classcode environment.

9.2. Additional footnotes to the title or authors’ names

The \thanks command may be used to produce additional footnotes to the title or authors’ names if required. Footnote symbols for this purpose should be used in the order: † (coded as \dagger), ‡ (\ddagger), § (\S), ¶ (\P), ‖ (\|), ‡‡ (\dagger\ddagger), §§ (\S\S), ¶¶ (\P\P), ‖‖ (\|\|).

Note that any footnotes to the main text will automatically be assigned the superscript symbols 1, 2, 3, etc. by the class file.\footnote{If preferred, the \texttt{endnotes} package may be used to set the notes at the end of your text, before the bibliography. The symbols will be changed to match the style of the journal if necessary by the typesetter.}
9.3. Lists

The \texttt{gIPE} class file provides numbered and unnumbered lists using the \texttt{enumerate} environment and bulleted lists using the \texttt{itemize} environment.

The enumerated list will number each list item with arabic numerals by default, e.g.

(1) first item
(2) second item
(3) third item

was produced by

\begin{enumerate}
  \item first item
  \item second item
  \item third item
\end{enumerate}

Alternative numbering styles can be achieved by inserting an optional argument in square brackets to each \texttt{item}, e.g. \texttt{item[(i)]} \texttt{first item} to create a list numbered with roman numerals.

Unnumbered lists can also be produced using the \texttt{enumerate} environment. For example,

First unnumbered item
Second unnumbered item
Third unnumbered item

was produced by

\begin{enumerate}
  \item First unnumbered item
  \item Second unnumbered item
  \item Third unnumbered item
\end{enumerate}

Bulleted lists are provided using the \texttt{itemize} environment. For example,

• First bulleted item
• Second bulleted item
• Third bulleted item

was produced by

\begin{itemize}
  \item First bulleted item
  \item Second bulleted item
  \item Third bulleted item
\end{itemize}

10. Some guidelines for using standard features

The following notes are intended to help you achieve the best effects with the \texttt{gIPE2e} class file.
10.1. **Sections**

\LaTeX{} provides five levels of section heading, all of which are defined in the `gIPE2e` class file:

(A) `\section`

(B) `\subsection`

(C) `\subsubsection`

(D) `\paragraph`

(E) `\subparagraph`

Section, subsection, subsubsection and paragraph headings are numbered automatically. If you need additional text styles in the headings, see the examples given in Section 11.

10.2. **Illustrations (figures)**

See ‘Instructions for Authors’ in the journal’s home page on the Taylor & Francis website for how to submit artwork (note that requests to supply figures and tables separately from text are for the benefit of authors using Microsoft Word; authors using \LaTeX{} may incorporate these at the appropriate locations). The source files of any illustrations will be required. Authors should ensure that their figures are suitable (in terms of lettering size, etc.) for the reductions they intend.

The `gIPE` class file will cope with most positioning of your illustrations and you should not normally need to use the optional placement specifiers of the `figure` environment. Figure captions should appear below the figure itself, therefore the `\caption` command should appear after the figure. For example, Figure 12 with caption and sub-captions is produced using the following commands:

\begin{verbatim}
\begin{figure}
\begin{center}
\subfigure[An example of an individual figure sub-caption.]{\resizebox*{5cm}{!}{\includegraphics{senu_gr1.eps}}}\hspace{5pt}
\subfigure[A slightly shorter sub-caption.]{\resizebox*{5cm}{!}{\includegraphics{senu_gr2.eps}}}
\caption{Example of a two-part figure with individual sub-captions showing that captions are flush left and justified if greater than one line of text, otherwise centred below the figure.}
\label{sample-figure}
\end{center}
\end{figure}
\end{verbatim}

The `\subfigure{}` and `\includegraphics{}` commands require subfigure.sty and graphicx.sty. The former is called in the preamble of the `gIPEguide.tex` file (in order to allow your choice of an alternative package if preferred) and the latter by the `gIPE2e` class file; both are included with the \LaTeX{} style guide package for this journal for convenience. Please supply any additional figure macros you use with your article in the preamble of your .tex file before `\begin{document}`.

To ensure that figures are correctly numbered automatically, the `\label{}` command should be inserted just after the `\caption{}` command, or in its argument.

The `epstopdf` package can be used to incorporate Encapsulated PostScript (.eps) illustrations when using PDF\LaTeX{}, etc. Please provide the original .eps source files rather than the generated PDF images of those illustrations for production purposes.
Figure 12. Example of a two-part figure with individual sub-captions showing that captions are flush left and justified if greater than one line of text, otherwise centred below the figure.

Table 2. Example of a table showing that its caption is as wide as the table itself and justified.

| Class   | $\gamma_1$ | $\gamma_2$ | $\langle \gamma \rangle$ | $G$ | $|f|$ | $\theta_c$ |
|---------|-------------|-------------|--------------------------|-----|------|----------|
| BL Lacs | 5           | 36          | 7                        | $-4.0$ | $1.0 \times 10^{-2}$ | $10^\circ$ |
| FSRQs   | 5           | 40          | 11                       | $-2.3$ | $0.5 \times 10^{-2}$ | $14^\circ$ |

*aThis footnote shows what footnote symbols to use.

*bThis footnote shows the text turning over when a long footnote is added.

10.3. Tables

The \textit{gIPE} class file will cope with most positioning of your tables and you should not normally need to use the optional placement specifiers of the \texttt{table} environment.

The \texttt{tabular} environment can be used as illustrated here to produce tables with appropriately spaced single thick and thin horizontal rules, which are allowed, if desired. Thick rules should be used at the head and foot only, and thin rules elsewhere as appropriate. Commands to redefine quantities such as \texttt{\textbackslash arraystretch} should be omitted.

The table caption appears above the body of the table in \textit{gIPE} style, therefore the \texttt{\textbackslash tbl} command should be used before the body of the table. For example, Table 2 is produced using the following commands. Note that \texttt{\textbackslash rm} will produce a roman character in math mode. There are also \texttt{\textbackslash bf} and \texttt{\textbackslash it}, which produce bold face and text italic in math mode.

\begin{verbatim}
\begin{table}
  \tbl{Example of a table showing that its caption is as wide as the table itself and justified.}
  \begin{tabular}{lcccccc}
    \hline
    Class & $\gamma_1$ & $\gamma_2$ & $\langle \gamma \rangle$ & $G$ & $|f|$ & $\theta_c$ \\
    \hline
    BL Lacs & 5 & 36 & 7 & $-4.0$ & $1.0 \times 10^{-2}$ & $10^\circ$ \\
    FSRQs & 5 & 40 & 11 & $-2.3$ & $0.5 \times 10^{-2}$ & $14^\circ$ \\
    \hline
  \end{tabular}
\end{table}
\end{verbatim}
To ensure that tables are correctly numbered automatically, the \texttt{\label{}} command should be inserted just before \texttt{\end{table}}.

Tables produced using the \texttt{booktabs} package for typesetting tables are also compatible with the \texttt{gIPE} class file.

### 10.4. Landscape pages

If a table or illustration is too wide to fit the measure it will need to be turned, along with its caption, through 90° anticlockwise. Landscape illustrations and/or tables can be produced using the \texttt{rotating} package, which is called by the \texttt{gIPE} class file. The following commands (for example) can be used to produce such pages.

\begin{verbatim}
\setcounter{figure}{0}
\begin{sidewaysfigure}
\centerline{\epsfbox{figname.eps}}
\caption{Example landscape figure caption.}
\label{landfig}
\end{sidewaysfigure}
\setcounter{table}{0}
\begin{sidewaystable}
\tbl{Example landscape table caption.}
{\begin{tabular}{@{}llllcll}
         .
         .
         .
\end{tabular}}\label{landtab}
\end{sidewaystable}
\end{verbatim}

Before any such float environment, use the \texttt{\setcounter} command as above to fix the numbering of the caption. Subsequent captions will then be automatically renumbered accordingly. The \texttt{\epsfbox{}} command requires \texttt{epsfig.sty}, which is called by the \texttt{gIPE2e} class file and included with the \texttt{LATeX} style guide package for this journal for convenience.

### 10.5. Theorem-like environments

A predefined \texttt{proof} environment is provided by the \texttt{amsthm} package (which is called by the \texttt{gIPE} class file), as follows:

\begin{proof}
More recent algorithms for solving the semidefinite programming relaxation are particularly efficient, because they explore the structure of the MAX-CUT problem. \hfill $\square$
\end{proof}

This was produced by simply typing:

\begin{verbatim}
\begin{proof}
More recent algorithms for solving the semidefinite programming
\end{verbatim}
relaxation are particularly efficient, because they explore the structure of the MAX-CUT problem.
\end{proof}

Other theorem-like environments (theorem, lemma, corollary, etc.) need to be defined as required, e.g. using \newtheorem{theorem}{Theorem}[section] in the preamble of your .tex file before \begin{document}. Theorem-like structures in gIPE are generally numbered as per the following examples:

**Theorem 10.1** More recent algorithms for solving the semidefinite programming relaxation are particularly efficient, because they explore the structure of the MAX-CUT problem.

**Lemma 10.2** More recent algorithms for solving the semidefinite programming relaxation are particularly efficient, because they explore the structure of the MAX-CUT problem.

**Corollary 10.3** More recent algorithms for solving the semidefinite programming relaxation are particularly efficient, because they explore the structure of the MAX-CUT problem.

**Proposition 10.4** More recent algorithms for solving the semidefinite programming relaxation are particularly efficient, because they explore the structure of the MAX-CUT problem.

**Definition 10.5** More recent algorithms for solving the semidefinite programming relaxation are particularly efficient, because they explore the structure of the MAX-CUT problem.

**Remark 10.6** More recent algorithms for solving the semidefinite programming relaxation are particularly efficient, because they explore the structure of the MAX-CUT problem.

These were defined as shown in detail in the preamble of the gIPEguide.tex file, and produced by typing, for example:

\begin{theorem}
More recent algorithms for solving the semidefinite programming relaxation are particularly efficient, because they explore the structure of the MAX-CUT problem.
\end{theorem}

The format of the text in these environments may be changed if necessary to match the style of the journal by the typesetter during preparation of your proofs.

10.6. **Typesetting mathematics**

10.6.1. **Displayed mathematics**

The gIPE class file will set displayed mathematics centred on the measure without equation numbers if the \LaTeX standard commands open (\[) and close (\]) square brackets
are used as delimiters. The equation
\[
\sum_{i=1}^{p} \lambda_i = \text{trace}(S) \quad i \in \mathbb{R}
\]
was typeset using the commands
\[
\sum_{i=1}^{p} \lambda_i = \text{trace}(S) \quad i \in \mathbb{R}
\]

For those of your equations that you wish to be automatically numbered sequentially throughout the text, use the \texttt{equation} environment, e.g.
\[
\sum_{i=1}^{p} \lambda_i = \text{trace}(S) \quad i \in \mathbb{R} \tag{16}
\]
was typeset using the commands
\[
\begin{equation}
\sum_{i=1}^{p} \lambda_i = \text{trace}(S) \quad i \in \mathbb{R}
\end{equation}
\]

Part numbers for sets of equations may be generated using the \texttt{subequations} environment, e.g.
\[
\varepsilon \rho w_{tt}(s,t) = N[w_{s}(s,t),w_{st}(s,t)]_{s}, \tag{17a}
\]
\[
w_{tt}(1,t) + N[w_{s}(1,t),w_{st}(1,t)] = 0, \tag{17b}
\]
which was generated using the commands
\[
\begin{subequations}
\begin{equation}
\varepsilon \rho w_{tt}(s,t) = N[w_{s}(s,t),w_{st}(s,t)]_{s}, \tag{subeqnpart}
\end{equation}
\begin{equation}
w_{tt}(1,t) + N[w_{s}(1,t),w_{st}(1,t)] = 0,
\end{equation}
\end{subequations}
\]

This is made possible by the \texttt{subeqn} package, which is called by the class file. If you put the \texttt{\label{}} just after the \texttt{\begin{subequations}} line, references will be to the collection of equations, \texttt{\(17\)} in the example above. Or, like the example code above, you can reference each equation individually – e.g. \texttt{\(17a\)}.
Displayed mathematics should be given end-of-line punctuation appropriate to the running text sentence of which it forms a part, if required.

10.6.2. **Bold math italic symbols**

To get bold math italic you can use \texttt{\textbf{bm}}, which works for all sizes, e.g.

\begin{equation}
\langle\alpha({\sf L})[RM(\{X_y + s_t\}) - RM(\{x_y\})]^2 \rangle
\end{equation}

Note that subscript, superscript, subscript to subscript, etc. sizes will take care of themselves and are italic, not bold, unless coded individually. \texttt{bm} produces the same effect as \texttt{\textbf{boldmath}}. \texttt{\textsf{sf}} allows upright sans serif fonts to be created in math mode by using the control sequence ‘\texttt{sf}’.

10.6.3. **Bold Greek**

Bold lowercase as well as uppercase Greek characters can be obtained by \texttt{\textbf{bm \gamma}}, which gives γ, and \texttt{\textbf{bm \Gamma}}, which gives Γ.

10.6.4. **Upright lowercase Greek characters and the upright partial derivative sign**

Upright lowercase Greek characters can be obtained with the \texttt{qIPE} class file by inserting the letter ‘u’ in the control code for the character, e.g. \texttt{\textmu} and \texttt{\upi} produce µ (used, for example, in the symbol for the unit microns – µm) and π (the ratio of the circumference to the diameter of a circle). Similarly, the control code for the upright partial derivative \texttt{\partial} is \texttt{\partial}.

10.7. **Acknowledgements**

An unnumbered section, e.g. \texttt{\section*{Acknowledgement(s)}}, should be used for thanks, etc. and included in the non-anonymous version before any Notes or References section.

10.8. **Disclosure**

An unnumbered section, e.g. \texttt{\section*{Disclosure}}, should be used to state any potential conflict of interest and included in the non-anonymous version before the References section, after any Acknowledgements and before any Funding statement.
10.9. **Funding**

An unnumbered section, e.g. \texttt{\section*{Funding}}, should be used for grant details, etc. and included in the non-anonymous version before any Notes or References section.

10.10. **Notes**

An unnumbered ‘Notes’ section may be placed before the References section (if using the \texttt{endnotes} package, use the command \texttt{\theendnotes} where the notes are to appear instead of creating a \texttt{\section}).

10.11. **Supplemental material**

Supplemental material should be referenced within your article where appropriate. An unnumbered section, e.g. \texttt{\section*{Supplemental material}}, detailing the supplemental material available should be placed immediately before the list of references, and should include a brief description of each supplemental file.

10.12. **References**

10.12.1. References cited in the text

References should be cited in accordance with US National Library of Medicine (NLM) style (please refer to the style guide in the journal’s Instructions for Authors for details). References are cited in the text by a number in square brackets (e.g. [1], [2,4,10], [11–15], not [11]–[15]), in the order in which they first appear.

Each bibliographical entry has a key, which is assigned by the author and used to refer to that entry in the text. In this document, the key \texttt{neu83} in the citation form \texttt{\cite{neu83}} produces ‘[1]’, and the keys \texttt{ed84} and \texttt{aiex00} in the citation form \texttt{\cite{ed84,aiex00}} produce ‘[2, 3]’. The citation for a range of bibliographic entries (e.g. ‘[4–11]’) will automatically be produced by \texttt{\cite{Eri1984,glov00,hk97,glov86,Agu95,Holl04,Shak78,Mil93}}. Optional notes may be included at the end of a citation by the use of square brackets, e.g. \texttt{\cite[see][p.73–77]{cwm73}} produces ‘[see 12, p.73–77]’.

10.12.2. The list of references

References should be listed at the end of the main text in the order in which they are first cited in the text. The following list shows some references prepared in the style of the journal.

**References**


This was produced by typing:

\begin{thebibliography}{99}

\bibitem{neu83} Neumann M. Parallel GRASP with path-relinking for job shop scheduling. 

\bibitem{ed84} Edwards DMF, McDonald IR. Positive bases in numerical optimization. 


\bibitem{Eri1984} Ericsson KA, Simon HA. Protocol analysis: verbal reports as data. 


\end{thebibliography}


\bibitem{Agu95} Agutter AJ. The linguistic significance of current British slang [unpublished doctoral dissertation]. Edinburgh (UK): Edinburgh University; 1995.


\bibitem{Mil93} Miller ME. The interactive tester (version 4.0) [computer software]. Westminster (CA): Psytek Services; 1993.


\end{thebibliography}

Each entry takes the form:
\bibitem{key} Bibliography entry

where ‘key’ is the tag that is to be used as an argument for the \cite{} commands in the text of the article and Bibliography entry is the material that is to appear in the list of references, suitably formatted.

Instead of typing the bibliography by hand, you may prefer to create the list of references using a \texttt{Bib}\TeX{} database. Include the lines
\bibliographystyle{gIPE}
\bibliography{gIPEGuide}
10.13. Appendices

Any appendices should be placed after the list of references, beginning with the command \appendices followed by the command \section for each appendix title, e.g.

\appendices
\section{This is the title of the first appendix}
\section{This is the title of the second appendix}

produces:

Appendix A. This is the title of the first appendix
Appendix B. This is the title of the second appendix

Subsections, equations, figures, tables, etc. within appendices will then be automatically numbered as appropriate.

11. Example of a section heading including SMALL CAPS, italic, and bold Greek such as κ

The following code shows how to achieve this section heading:

\section{Example of a section heading including \fontencoding{T1}\textsc{small caps}, \textit{italic}, and bold Greek such as $\bm{\kappa}$}\label{headings}

12. gIPE journal style

The notes given here relate to common style errors found in manuscripts, but are not intended to be exhaustive.

12.1. Hyphens, en rules, em rules and minus signs

(i) Hyphens (one dash in \TeX/\LaTeX). \textit{gIPE} uses hyphens for compound adjectives (e.g. low-density gas, least-squares fit, two-component model) but not for complex units or ranges, which could become cumbersome (e.g. 15 km s$^{-1}$ feature, 100–200 µm observations).

(ii) en rules (two dashes in \TeX/\LaTeX). These are used (a) to denote a range (e.g. 1.6–2.2 µm); (b) to denote the joining of two words of equal standing (e.g. Kolmogorov–Smirnov test, Herbig–Haro object); (c) with spaces, as an alternative to parentheses (e.g. ‘the results – assuming no temperature gradient – are indicative of . . . ’).

(iii) The em rule (three dashes in \TeX/\LaTeX) has no specified use in \textit{gIPE}.
(iv) The minus sign is produced automatically in math mode by the use of a single dash, e.g.

\[ y_i \in \{-1, 1\} \quad \forall i \in V, \tag{19} \]

where $|\!-\!V| = A^2 + B^2$.

is produced by

\begin{equation}
\begin{split}
y_i \in \{-1, 1\} & \quad \forall i \in V, \\
\end{split}
\end{equation}

\noindent where $|\!-\!V| = A^2 + B^2$.

\section{References}

It is important to use the correct reference style, details of which can be found in Section 10.12.

\section{Maths fonts}

Scalar variables should be mediumface italic (e.g. $s$ for speed); vectors should be bold italic (e.g. $v$ for velocity); matrices should be bold roman (upright) (e.g. $A$), and tensors should be bold upright sans serif (e.g. $L$). Differential $d$, partial differential $\partial$, complex $i$, exponential $e$, superscript $T$ for ‘transpose’, $\sin$, $\cos$, $\tan$, $\log$, etc., should all be roman. Openface, or ‘blackboard’, fonts can be used, for example, for the integers $\mathbb{Z}$ and the reals $\mathbb{R}$. Sub/superscripts that are physical variables should be italic, while those that are labels should be roman (e.g. $C_p$, $T_{\text{eff}}$).

\section{Troubleshooting}

Authors may from time to time encounter problems with the preparation of a paper using $\LaTeX$. The appropriate action to take will depend on the nature of the problem – the following is intended as a guide.

(i) If the problem is with $\LaTeX$ itself, rather than with the actual macros, please refer to an appropriate handbook for initial advice. If the solution cannot be found, or if you suspect that the problem lies with the macros, then please contact Taylor & Francis for assistance (latex.helpdesk@tandf.co.uk).

(ii) Problems with page make-up (e.g. large spaces between paragraphs, above headings, or below figures; figures/tables appearing out of order): please do not attempt to remedy these using ‘hard’ page make-up commands – the typesetter will deal with such problems. (You may, if you wish, draw attention to particular problems when submitting the final version of your paper.)

(iii) If a required font is not available at your site, allow $\TeX$ to substitute the font and specify which font you require in a covering letter accompanying your file(s).
14. Fixes for coding problems

This guide has been designed to minimize the need for user-defined macros to create special symbols. Authors are urged, wherever possible, to use the following coding rather than to create their own. This will minimize the danger of author-defined macros being accidentally ‘overridden’ when the paper is typeset (see Section 10.6, ‘Typesetting mathematics’). In cases where it is essential to create your own macros, these should be displayed in the preamble of your .tex file before \begin{document}.

(i) Fonts in section headings and paper titles. The following are examples of styles that sometimes prove difficult to code.

**Paper titles:**

Generalized Flory theory at $\delta > 50^\circ$

is produced by

\title{Generalized Flory theory at $\bm{\delta > 50^\circ}$}

Ion--ion correlations in H\textsc{ii} regions

is produced by

\title{Ion--ion correlations in H\textsc{ii} regions}

(ii) en rules, em rules, hyphens and minus signs (see Section 12.1 for correct usage). To create the correct symbols in the sentence

The high-resolution observations were made along a line at an angle of $-15^\circ$ (East from North) from the axis of the jet -- which runs North--South

you would use the following code:

\text{The high-resolution observations were made along a line at an angle of $-15^\circ$ (East from North) from the axis of the jet -- which runs North--South}

(iii) Fonts in superscripts and subscripts. Subscripts and superscripts will automatically come out in the correct font and size in a math environment (e.g. enclosed by ‘$’ delimiters in running text or within \[...\] or the ‘equation’ environment for displayed equations). You can create the output $k_x$ by typing $\bm{k_x}$. If the subscripts or superscripts need to be other than italic, they should be coded individually – see (vi) below.

(iv) Calligraphic letters (**UPPER CASE ONLY**). Normal calligraphic can be produced with \texttt{\mathcal} as usual (in math mode).

(v) Automatic scaling of brackets. The codes \texttt{\left} and \texttt{\right} should be used to scale brackets automatically to fit the equation being set. For example, to get

$$v = x \left( \frac{N + 2}{N} \right)$$

use the code

36
\[ v = x \left( \frac{N+2}{N} \right) \]

(vi) Roman font in equations. It is often necessary to make some symbols roman in an equation (e.g. units, non-variable subscripts). For example, to get

\[
\sigma \simeq \left( \frac{r}{13 \, h^{-1} \, \text{Mpc}} \right)^{-0.9}, \quad \omega = \frac{N - N_s}{N_R}
\]

use the code

\[
\sigma \simeq \left( r/13^{\rm h^{-1}\text{Mpc}} \right)^{-0.9}, \quad \omega = \frac{N-N_s}{N_R}
\]

The \texttt{SIunits} package of macros for typesetting units is also compatible with the \texttt{gIPE2e} class file.

15. **Obtaining the gIPE2e class file**

15.1. *Via the Taylor & Francis website*

This Guide for Authors and the \texttt{gIPE2e} class file may be obtained via the Instructions for Authors on the Taylor & Francis homepage for the journal.

Please note that the class file calls up the following open-source \LaTeX{} packages, which will, for convenience, unpack with the downloaded Guide for Authors and class file: \texttt{amsbsy.sty}; \texttt{amsfonts.sty}; \texttt{amsmath.sty}; \texttt{amssymb.sty}; \texttt{epsfig.sty}; \texttt{graphicx.sty}; \texttt{natbib.sty}; \texttt{rotating.sty}. The Guide for Authors calls for subfigure.sty, which is also supplied for convenience.

15.2. *Via e-mail*

This Guide for Authors, the class file and the associated open-source \LaTeX{} packages are also available by e-mail. Requests should be addressed to \texttt{latex.helpdesk@tandf.co.uk} clearly stating for which journal you require the Guide for Authors and/or class file.