Part 2 – Introduction to Microlocal Analysis

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August 2nd, 2010
Outline

PART II

• Pseudodifferential ($\psi$DOs) and Fourier Integral Operators (FIOs)
  – Definitions
  – Motivation for the study of FIOs and $\psi$DOs
  – Symbols/Amplitudes/Filters

• Propagation of singularities
  – Singular support
  – Wavefront Sets
  – Method of Stationary Phase
  – Effect of $\psi$DOs and FIOs on Singular Support and Wavefront Sets

• Inversion of FIOs
  – Canonical Relations
  – Construction of Filters

• Conclusion
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The Radon Transform

• Definition –

\[ Rf(\theta, p) = \int_{x \cdot \theta = p} f(x) \, dx. \]

• Fourier slice theorem –

\[ Rf(\theta, p) = \frac{1}{2\pi} \int e^{i\omega(x \cdot \theta - p)} f(x) \, dx \, d\omega. \]

• Radon transform is an FIO
ψDOs and FIOs

• General integral representations:

- ψDOs:

\[ \mathcal{P}[f](x) = \int e^{i(x-y)\cdot \xi} A(x, y, \xi) f(y) dy d\xi \]

- FIOs:

\[ \mathcal{F}[f](x) = \int e^{i\varphi(x,y,\xi)} A(x, y, \xi) f(y) dy d\xi \]

Symbols/Amplitudes/Filters

Phase function

Input variable

Frequency variable

Output variable
Motivation for study of FIOs

• Measured data in several imaging problems in the form of $\mathcal{F}[f]$
• Exact determination of $f$ often not possible.
• Inhomogeneities in the medium carry much information.
• Reconstruct the inhomogeneities (modeled as singularities or sharp changes of $f$).
• FIOs propagate singularities in specific ways.
• Inversion of FIOs reconstructs the singularities of the medium.
Symbol of a PDE

- Linear partial differential operators:

\[ \mathcal{P}(x, D) = \sum_{|\nu| \leq m} a_\nu(x) D_\nu^\nu. \]

\[ D_\nu^\nu = (-i)^{\nu_1 + \cdots + \nu_n} \frac{\partial^{\nu_1}}{\partial x_1^{\nu_1}} \cdots \frac{\partial^{\nu_n}}{\partial x_n^{\nu_n}}. \]

\[ \mathcal{P}[f](x) = \frac{1}{(2\pi)^n} \int e^{ix \cdot \xi} \left( \sum_{|\nu| \leq m} a_\nu(x) \xi^\nu \right) \hat{f}(\xi) d\xi \]

\[ = \frac{1}{(2\pi)^n} \int e^{i(x-y) \cdot \xi} A(x, \xi) f(y) dy d\xi \]
Estimates for the Symbol

- \( A(x, \xi) \) - Polynomial in \( \xi \)

- Let \( \alpha \) and \( \beta \) be any multi-indexes. For \( x \) in a bounded subset

\[
\left| \partial_\xi^\alpha \partial_x^\beta A(x, \xi) \right| \leq C(1 + |\xi|)^{m-|\alpha|}
\]

- Differentiation with respect to \( x \) does not change the degree of the polynomial

- Differentiation with respect to \( \xi \) lowers the degree of the polynomial.
Symbols for $\psi$DOs and FIOs

• Consider $A(x, \xi)$ that satisfy the estimate

$$|\partial_\xi^\alpha \partial_x^\beta A(x, \xi)| \leq C(1 + |\xi|)^{m-|\alpha|}$$

• Behave like polynomials or inverse of polynomials in $\xi$ as $|\xi| \to \infty$

• $A(x, \xi)$ grows or decays in powers of $|\xi|$. Differentiation with respect to $\xi$ lowers the order of growth or increases the order of decay.

• Symbol of order $m$. The order $m$ need not be an integer.

• Needed for the method of stationary phase/high frequency approximations.
Symbols - Examples

- $|\xi|^2$
- $\frac{1}{1 + |\xi|^2}$
- $|\xi|^{1/2}$
- $e^{ix \cdot \xi}$ - Not a symbol
- $\frac{\chi(x)}{(1 + |\xi|^2)^5}$

where $\chi(x)$ is a smooth function.
Pseudodifferential Operators

• A pseudodifferential operator $\mathcal{P}$ of order $m$

$$\mathcal{P}[f](x) = \int e^{i(x-y)\cdot \xi} A(x, y, \xi) f(y) dy \, d\xi$$

• $A(x, y, \xi)$ satisfies the amplitude estimate

$$|\partial_{\xi}^\alpha \partial_x^\beta \partial_y^\gamma A(x, y, \xi)| \leq C (1 + |\xi|)^{m-|\alpha|}$$

• $e^{i(x-y)\cdot \xi}$ phase term, highly oscillating
Pseudodifferential Operators - Examples

• Linear partial differential operators

• Operator \( \sqrt{\Delta} \)

\[
\sqrt{\Delta}[f](x) = \int e^{i(x-y) \cdot \xi} |\xi| f(y) dy d\xi
\]

• Time-invariant filters in signal processing

\[
\mathcal{A}[u](t) = [u * f](t)
\]

• Time-invariant filters –

\[
\mathcal{A}[u](t) = \int e^{i(t-s)\omega} \hat{f}(\omega) u(s) ds d\omega
\]

Approx. by a rational polynomial of degree m
Pseudodifferential Operators - Examples

• If $\mathcal{B}$ is a time-varying convolution:

$$\mathcal{B}[u](t) = \int f(t, t - s)u(s) ds.$$ 

• In the Fourier domain – Approx. by a rational polynomial with time-varying coefficients of order $m$

$$\mathcal{B}[u](t) = \int e^{i(t-s)\omega} \hat{f}_2(t, \omega)u(s) ds d\omega$$

• This is a pseudodifferential operator.
Fourier Integral Operators

• Operators more general than DOs.
• More general phase function.
• Phase function retains the essential properties of that of PseudoDOs.
• One important difference: Dimensions of $x$, $y$ and $\xi$ can be all different
• Example: Radon transform:

$$Rf(\theta, p) = \frac{1}{2\pi} \int e^{i\omega(x\cdot\theta - p)} f(x) \, dx \, d\omega.$$
Phase Function of FIOs

- Phase function \( \varphi(x, y, \xi) \) should satisfy the following properties:
  - Real-valued and smooth in \( x, y \) and \( \xi \).
  - Homogeneous of order 1 in \( \xi \)
    
    \[
    \varphi(x, y, \lambda \xi) = \lambda \varphi(x, y, \xi) \quad \lambda > 0
    \]
  - \( [\partial_x \varphi, \partial_\xi \varphi]^T \) and \( [\partial_y \varphi, \partial_\xi \varphi]^T \) are non-zero vectors for \( x, y, \xi \) with \( \xi \neq 0 \).
Fourier Integral Operators

- Fourier integral operators:

\[ \mathcal{F}[f](x) = \int e^{i \varphi(x, y, \xi)} A(x, y, \xi) f(y) dy d\xi \]

- \( \varphi(x, y, \xi) \) satisfies properties of a phase function

- \( A(x, y, \xi) \) satisfies estimates:

\[ |\partial_{\xi}^{\alpha} \partial_{x}^{\beta} \partial_{y}^{\gamma} A(x, y, \xi)| \leq C (1 + |\xi|)^{m-|\alpha|} \]
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Singular Support

- The set of points where a function is not smooth
- Not smooth – not infinitely differentiable
- The set of non-smooth points – singular support
- Notation: \( \text{singsupp}(f) \)
- Singular support of \( \delta \) is \( \{0\} \)
Singular Support - Examples

- Consider the square $S$ on the plane:

\[ \{(x, y) : |x| \leq 1 \text{ and } |y| \leq 1\} \]

- Let $u$ be the function:

\[ u(x, y) = \begin{cases} 
1 & \text{if } (x, y) \in S; \\
0 & \text{otherwise.} 
\end{cases} \]

- $\text{singsupp}(u)$ is the boundary of the square.
Singulärer Support - Beispiele

- Betrachten Sie die Funktion auf dem Raume:

\[ v(x) = \begin{cases} 1 - (x^2 + y^2)^{3/2} & \text{für } \sqrt{x^2 + y^2} \leq 1; \\ 0 & \text{sonst}. \end{cases} \]

\( \text{singsupp}(v) \) ist \( \{(x, y) : x^2 + y^2 = 1\} \).

- Erste Ableitungen \( \partial_x v \) und \( \partial_y v \) existieren.

- Höhere Ableitungen existieren nicht an Punkten auf dem Einheitskreis.
Wavefront Sets

• Wavefront sets – Directions of singularity at the location of a singularity
• Smoothness of a function corresponds to decay of its Fourier transform
• Localize a function near the location of a singularity and consider the localized Fourier transform
• Microlocal information – Consider directions in which the localized Fourier transform does not decay
Wavefront Sets

Singular directions are characterized based on the behavior of the localized Fourier transform.
Wavefront Sets

Singular directions are characterized based on the behavior of the localized Fourier transform.
Wavefront Sets

- $x_0$ a location of singularity for a function $f$ and consider a direction $\xi_0$

- $f$ is microlocally smooth near $(x_0, \xi_0)$ if the localized Fourier transform decays rapidly

- The complement of the set of points $(x_0, \xi_0)$ near which $f$ is microlocally smooth is called the wavefront set of $f$

- Denoted $WF(f)$
Wavefront Sets-Examples

Left figure – Circle in red is the singular support of $f$
Right figure- Arrows indicate the wavefront set directions at points belonging to $\text{sing supp}(f)$
Wavefront Sets-Examples

Left figure – Square in red is the singular support of $f$
Right figure- Arrows indicate the wavefront set directions at points belonging to $\text{singsupp}(f)$
Propogation of Singularities

Goal

Understand how a PseudoDO or an FIO carries the singularities of $f$ to $\mathcal{P}[f]$ or $\mathcal{F}[f]$.

This singularity does not appear in the output space.

This singularity appears in three places in the output space.
Method of Stationary Phase

• Expansion for certain oscillatory integrals of the form

\[ I(\lambda) = \int e^{i\lambda \varphi(x)} u(x) \, dx \]

• Conditions:
  \[ \partial_x \varphi(x_0) = 0 \]
  - \( x_0 \) a critical point:
  - Critical point is non-degenerate:
  \[ \det(\partial^2_{x_i x_j} \varphi(x_0)) \neq 0 \]
Method of Stationary Phase

Near the origin the function stays positive. Hence non-zero contribution to the integral.
Method of Stationary Phase

Asymptotic expansion for the integral –

\[ \int e^{i\lambda \varphi(x)} u(x) dx = e^{i\lambda \varphi(x_0)} \left( \sum_{|\alpha|=0}^{N-1} (A_2\alpha(D)u)(x_0) \lambda^{-\alpha-n/2} \right) + \]

\[ + \mathcal{O}(\lambda^{-N-n/2}). \]

The first term in the asymptotic expansion

\[ A_0 = \frac{(2\pi)^{n/2} e^{i\frac{\pi}{4} \text{sgn}(\partial_{x_i x_j} \varphi(x_0))}}{|\det \partial_{x_i x_j} \varphi(x_0)|^{1/2}}. \]

Special case of the stationary phase method

\[ \int e^{i\lambda \varphi(x)} u(x) dx = \frac{(2\pi/\lambda)^{n/2} e^{i\frac{\pi}{4} \text{sgn}(\partial_{x_i x_j} \varphi(x_0))}}{|\det \partial_{x_i x_j} \varphi(x_0)|^{1/2}} e^{i\lambda \varphi(x_0)} u(x_0) + \]

\[ + \mathcal{O}(\lambda^{-1-n/2}). \]

\( n \) - Dimension of the integral

\( \text{sgn} \) – Difference of the number of positive and negative eigenvalues
Method of Stationary Phase - Application

Relation between amplitudes and symbols of PseudoDOs.

\[ \mathcal{P}[f](x) = \int e^{i(x-y) \cdot \xi} A(x, y, \xi) f(y) dy d\xi \]

\[ \mathcal{P}[f](x) = \int e^{ix \cdot \eta} (e^{i(x-y) \cdot (\xi-\eta)} A(x, y, \xi) dy d\xi) \hat{f}(\eta) d\eta. \]

Using MSP:

\[ \tilde{A}(x, \eta) = \sum_{\alpha \geq 0} \frac{1}{(i)^{\alpha} |\alpha| \alpha!} \partial^\alpha_y \partial^\alpha_\xi A(x, y, \xi) |_{y=x, \xi=\eta}. \]
Effect of $\psi$DOs on Singular Support

• Goal: Understand the effect of $\psi$DO on the singular support of a function

• What is the relation between $\text{singsupp}(\mathcal{P}[f])$ and $\text{singsupp}(f)$?

• Important tools:
  – Amplitude estimates
  – Method of stationary phase
Effect of $\psi$DOs on Singular Support

- Kernel representation for a $\psi$DO:

\[ P[f](x) = \int e^{i(x-y)\cdot\xi} A(x, y, \xi) f(y) dy d\xi \]
\[ = \int P(x, y) f(y) dy \]
\[ P(x, y) = \int e^{i(x-y)\cdot\xi} A(x, y, \xi) d\xi \]

- Use the method of stationary phase for this kernel
Effect of $\psi$DOs on Singular Support

• Derivative with respect to the frequency variable:
  $$\partial_\xi ((x - y) \cdot \xi) = x - y$$

• For $x \neq y$ no stationary points

• Kernel is smooth in the complement of the set:
  $$\{(x, y) : x = y\}$$

• The relation:
  $$\text{singsupp}(\mathcal{P}[f]) \subset \text{singsupp}(f)$$
Effect of FIOs on Singular Support

- Kernel representation for FIOs –

\[
\mathcal{F}[f](x) = \int e^{i\varphi(x,y,\xi)} A(x, y, \xi) f(y) dy d\xi = \int F(x, y) u(y) dy
\]

- Kernel of FIO –

\[
F(x, y) = \int e^{i\varphi(x,y,\xi)} A(x, y, \xi) d\xi
\]
Effect of FIOs on Singular Support

- Use the method of stationary phase: The major contribution to $F(x, y)$ comes from the set of points

$$\left\{ (x, y) : \text{there exists } \xi \text{ with } \partial_\xi \varphi(x, y, \xi) = 0 \right\}.$$ 

- The relationship:

$$\text{singsupp}(\mathcal{F}[f]) \subset \left\{ x : \text{there exists } y \in \text{singsupp}(f) \right\}$$

and $\xi \neq 0$ such that $\partial_\xi \varphi(x, y, \xi) = 0$. 
Effect of FIOs on Singular Support-Example

- Forward model for SAI:

\[
\mathcal{F}_{MS}[T](s, t) = \int e^{-i\omega(t-\frac{2}{c_0}|x-\gamma(s)|)} A(x, s, \omega) T(x) dx d\omega
\]

- Phase function:

\[
\omega(t-\frac{2}{c_0}|x-\gamma(s)|)
\]

- The singularities of the kernel lie in

\[
\left\{ ((s, t), x) : c_0 t = 2|x - \gamma(s)| \right\}
\]
Effect of FIOs on Singular Support-Example

- Based on the relation between singular supports: If a point $x_0 \in \text{singsupp}(T)$, the singularity at $x_0$ could propagate to (some or all of)

\[ \left\{ (s, t) : c_0 t = 2|x_0 - \gamma(s)| \right\}. \]

Obtain a precise description of how propagation of singularities occur.
Effect of $\psi$DOs and FIOs on Wavefront Sets

- Let $P$ be a PseudoDO

\[ WF(P[f]) \subset WF(f) \]

- Let $F$ be an FIO

\[ WF(F[f]) \subset \left\{ (x, \partial_x \varphi(x, y, \xi)) \text{ with } (x, y, \xi) \text{ such that } \partial_\xi \varphi(x, y, \xi) = 0 \text{ and } (y, -\partial_y \varphi(x, y, \xi)) \in WF(f) \right\} \]
SAI Example - Revisited

- Phase function in SAI:

\[ \varphi(s, t, x, \omega) = \omega(t - \frac{2}{c_0}|x - \gamma(s)|) . \]

\[ \varphi_\omega(s, t, x, \omega) = 0 \implies c_0 t = 2|x - \gamma(s)| . \]

\[ WF(F_{\text{MS}}[T]) \subset \left\{ (s, t, -\frac{2\omega}{c_0} \hat{x} - \gamma(s) \cdot \dot{\gamma}(s), -\omega) : c_0 t = 2|x - \gamma(s)| \text{ and } (x, (x - \gamma(s))_H) \in WF(T) \right\} . \]

Hat – Unit vector

Doppler

H denotes projection on to the ground.
SAI Example - Revisited

This wavefront set direction is not propagated to the data
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Canonical Relations

Recall

- Kernel for $\psi$DOs:

$$P(x, y) = \int e^{i(x-y)\cdot\xi} A(x, y, \xi) d\xi$$

- Kernel for FIOs:

$$F(x, y) = \int e^{i\varphi(x,y,\xi)} A(x, y, \xi) d\xi$$

Canonical Relations $\equiv$ Wavefront sets of the kernels of $\psi$DOs and FIOs
Canonical Relations

• Treat kernels as functions:
  - Wavefront set of the kernel of PseudoDO:

$$\mathcal{C}_{\Psi DO} : \text{Wavefront set of } WF(P) \subset \{(x, \xi, x, \xi) : \xi \neq 0\}.$$  

- Wavefront set of the kernel of FIO:

$$\mathcal{C}_{\text{FIO}} : \text{Wavefront set of } WF(F) \subset \left\{ (x, \partial_x \varphi(x, y, \xi), y, -\partial_y \varphi(x, y, \xi)) \right\} \text{ with } (x, y, \xi) \text{ such that } \partial_\xi \varphi(x, y, \xi) = 0.$$

Wavefront set of input data
Wavefront set of output data
Criticality condition in SPM
Canonical Relations

- The wavefront set relation for PseudoDOs reinterpreted as

\[ WF(\mathcal{P}[f]) \subset C_{\psi DO} \circ WF(f). \]

- The wavefront set relation for FIOs reinterpreted as

\[ WF(\mathcal{F}[f]) \subset C_{\text{FIO}} \circ WF(f). \]

- Interpreted as composition of relations
Composition of Relations

- If $C = \{(x, \xi, y, \eta)\}$ and $WF(u) = \{(y, \eta)\}$

$$C \circ WF(u) = \{(x, \xi) : \text{there exists } (y, \eta) \in WF(u) \text{ and } (x, \xi, y, \eta) \in C.\}$$

- If $C_1 = \{(x_1, \xi_1, y_1, \eta_1)\}$ and $C_2 = \{(y_2, \eta_2, x_2, \xi_2)\}$

$$C_1 \circ C_2 = \{(x_1, \xi_1, x_2, \xi_2) : \text{there exists } (y, \eta) \text{ such that } (x_1, \xi_1, y, \eta) \in C_1 \text{ and } (y, \eta, x_2, \xi_2) \in C_2\}.$$
Hörmander-Sato Lemma

• Consider two FIOs: \( \mathcal{F}_1 \) with canonical relation \( C^{\mathcal{F}_1} \) and \( \mathcal{F}_2 \) with canonical relation \( C^{\mathcal{F}_2} \).

• Hörmander-Sato Lemma:

\[
WF(\mathcal{F}_1 \mathcal{F}_2) \subset C^{\mathcal{F}_1} \circ C^{\mathcal{F}_2}
\]

• Analyze propagation of singularities or wavefront sets using this lemma
Inversion of FIOs

• Construct approximate inverse $\mathcal{K}$ of $\mathcal{F}$

• Choose $\mathcal{K}$ so that the location and orientation of the singularities are preserved

$$\mathcal{K}\mathcal{F} \approx I$$

• Microlocal Analysis $\Rightarrow$
  
  – If $\mathcal{K}\mathcal{F}$ is a pseudo-differential operator $\Rightarrow$ preserves location and orientation of singularities
  
  – Analyze propagation of singularities / wavefront sets

$$WF(\mathcal{K}\mathcal{F}) \subset C^\mathcal{K} \circ C^\mathcal{F}$$
Inversion of FIOs

• To choose $\mathcal{K}$ such that $\mathcal{K}\mathcal{F}$ is a PseudoDO:

\[ \mathcal{K} = \mathcal{F}^* \]

• $\mathcal{F}^*$ is the $L^2$-adjoint (backprojection operator).

• $\mathcal{F}^*\mathcal{F}$ is a PseudoDO. Reconstructs the singularities at the correct location and orientation.

• Will not reconstruct at the right strength. Design a filter to reconstruct singularities at the right strength.
Construction of Filters

- Consider a PseudoDO

\[ \mathcal{P}[f](x) = \int e^{i(x-y) \cdot \xi} A(x, \xi) f(y) dy \, d\xi \]

- Find another $\psi$DO

\[ \mathcal{Q}[g](z) = \int e^{i(z-w) \cdot \eta} B(z, \eta) g(w) dw \, d\eta \]

- Choose $\mathcal{Q}$ such that $\mathcal{P} \circ \mathcal{Q} \approx \text{Identity}$.
Construction of Filters

• Kernel representation for the composition:

\[ \mathcal{P} \circ \mathcal{Q}[f](x) = \int R(x, y) f(y) dy. \]

• Choose \( \mathcal{Q} \) such that

\[ R(x, y) \approx \delta(x - y). \]

• The composition in terms of symbols:

\[ \mathcal{P} \mathcal{Q}[f](x) = \int e^{ix \cdot \eta} \left( \int e^{i(x-y) \cdot (\xi - \eta)} A(x, \xi) B(y, \eta) dy d\xi \right) \hat{f}(\eta) d\eta. \]
Construction of Filters

• Using SPM:

\[ R(x, \eta) \sim \sum_{\alpha \geq 0} \frac{1}{(i)^{|\alpha|/\alpha!}} \frac{1}{\partial_\xi^\alpha A(x, \xi) \partial_y^\alpha B(y, \eta)} \bigg|_{y=x, \xi=\eta}. \]

• Set

\[ R(x, \eta) = 1 \]

• Let

\[ B(y, \eta) = \sum_{n=0}^{\infty} B_n(y, \eta) \]

where \( B_n \) are homogeneous of decreasing order in \( \eta \).
Construction of Filters

- Iteratively determine $B_n$

\[
B_0 = \frac{1}{A}
\]

\[
A \cdot B_1 + \sum_{|\alpha|=1} \frac{1}{i^\alpha} \partial_\xi^\alpha A \partial_x^\alpha B_0 = 0.
\]

\[
A \cdot B_2 + \sum_{|\alpha|=2} \frac{1}{i^{2\alpha} \alpha!} \partial_\xi^\alpha A \partial_x^\alpha B_0 + \sum_{|\alpha|=1} \frac{1}{i} \partial_\xi^\alpha A \partial_x^\alpha B_1 = 0.
\]

etc…
Conclusion

• Several wave-based imaging problems – Data models are FIOs
• Inhomogeneities of a medium modeled as singularities
• Singularities – Wavefront Set (location and orientation)
• Pseudodifferential operators and Fourier integral operators propagate wavefront sets in specific ways
• Back propagation – Reconstructions the singularities at the right location and orientation
• An appropriate choice of filter reconstructs the singularities at the right strength