Microlocal Analysis in Imaging

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Outline

• Part I – Motivations – 1:30-1:50

• Break – 1:50-2:00pm

• Part II – Basics of microlocal analysis – 2:00-3:00pm

• Break – 3:00-3:10pm

• Part III – Applications of microlocal analysis in imaging – 3:10-4:30pm
Table of Contents

PART I

• Motivations
• What is microlocal analysis?
• Mathematics of imaging
• Advantages of microlocal techniques
• Conclusion
Table of Contents

PART II

• Pseudodifferential and Fourier Integral Operators (FIOs)
• Singular Support and Wavefront sets
• Method of Stationary Phase
• Action of FIOs on Singular Support and Wavefront Sets
• Canonical Relations
• Inversion of FIOs
Table of Contents

PART III

• Synthetic Aperture Imaging (SAI) modalities
• SAI forward model
• SAI image formation via microlocal techniques
  – Bistatic SAI
  – Monostatic SAI
  – Multistatic SAI
  – Synthetic Aperture Hitchhiker
  – Doppler Synthetic Aperture Hitchhiker
• Conclusions
Part 1 – Introduction & Motivations

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Microlocal Analysis in Imaging

- Near ground sensors
- Close-in sensors
- Space-based sensors

Geological Monitoring
Ecological/Environmental monitoring
Surveillance/Security
Civil infrastructure monitoring
Weather monitoring
Human health monitoring
Problem Space and the Role of Microlocal Analysis

• Themes
  – Multiple heterogeneous mobile or stationary autonomous sensors
  – Sensors with adaptive transmit and receive parameters (arbitrary trajectories, waveforms etc.)
  – Operating in complex environments – multiple scattering, dynamically changing medium, clutter, noise etc.

• Approach
  – Physics-based and statistical modeling
  – Tomographic approach

• Microlocal analysis – An integral bridging role
Microlocal Analysis

• **Microlocal analysis** – Abstract mathematical theory of singularities, associated high frequency structures and *Fourier Integral Operators* (FIOs)

• Roots in physics…

• **Key concepts**: *Singularities or wavefront sets* → A mathematical description of “edges”, heterogeneities, or not so “well-behaving” parts of the medium

• FIO theory → A common framework for many wave-based imaging/sensing problems

• Can work in complex environments
Microlocal Analysis

• “Local”: refers to local processing:
  • Localizing a function near a point – Multiplying by a smooth non-vanishing function

• “Micro”: refers to directional analysis:
  • Analyzing the directional Fourier transform of the localized function

• Strong connection to adaptive signal processing & physics
History of Microlocal Analysis

- Term introduced by three Japanese mathematicians – Sato, Kawai and Kashiwara in the 1970s
- Developed from 1950s onwards
- Sato (Wolf prize ‘03) in 1969 was one of the first to analyze microlocal behavior of functions
- Hörmander (Fields medal ‘62, Wolf prize ‘88) introduced microlocal techniques in the context of partial differential equations (PDEs)
- Early roots in geometric optics
Microlocal Analysis in Imaging Problems

- Inversion of X-ray transforms – Katsevich
- Electrical impedance tomography – Sylvester and Uhlmann
- Reflection seismology – De Hoop, Symes, Bleistein, Petkov and Stoyanov and many others
- Synthetic aperture and inverse synthetic aperture imaging – Nolan, Cheney and Yazıcı
- Electron and thermo acoustic tomography – Quinto, Kuchment
Objectives

• Well-developed mathematical theory

• Introduce key concepts and techniques of microlocal analysis to engineering community

• Demonstrate its strong relevance to challenging sensing and imaging problems

• Establish the terminology and commonalities between signal processing and microlocal analysis

• Motivate wider applications of the theory in engineering problems
Sensing and Imaging

**Imaging** – Process of extracting *spatially resolved physical, biological or chemical property* of an object or medium by means of an *imaging probe* (wave, particle etc.); a *measurement system* and by subsequent *processing of measurements*.
## Probes, Physical Properties and Imaging Systems

<table>
<thead>
<tr>
<th>Probe</th>
<th>Parameter</th>
<th>System</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acoustic</td>
<td>Density, Compressibility</td>
<td>Seismic Imaging, Ultrasonic Imaging</td>
</tr>
<tr>
<td>Electrical</td>
<td>Conductivity, Permittivity</td>
<td>Electrical Impedance Tomography (EIT)</td>
</tr>
<tr>
<td>Magnetic</td>
<td>Permeability</td>
<td>Electromagnetic Induction (EMI)</td>
</tr>
<tr>
<td>Radiowave</td>
<td>Permittivity, Conductivity</td>
<td>Ground Penetrating Radar (GPR)</td>
</tr>
<tr>
<td>Microwave</td>
<td>Nuclear spin</td>
<td>Magnetic Resonance Imaging (MRI)</td>
</tr>
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<td>Microwave Imaging</td>
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<tr>
<td></td>
<td></td>
<td>Terahertz Imaging</td>
</tr>
<tr>
<td>Optical</td>
<td>Absorption coefficient, Refractive index, Fluorescence rate, Diffusion &amp; absorption coefficient</td>
<td>Infrared Imaging, Optical Imaging, Fluorescence Imaging, Diffuse Optical Tomography (DOT)</td>
</tr>
<tr>
<td>X-rays</td>
<td>Absorption coefficient</td>
<td>Computed Tomography (CT)</td>
</tr>
<tr>
<td>γ-rays</td>
<td>Density of radio nucleids</td>
<td>Nuclear medicine</td>
</tr>
<tr>
<td>Particles</td>
<td>Flux of chemical emissions, Flux of nuclear emissions, Scattering cross-section, Rate of secondary emissions</td>
<td>Chemical sensing, Nuclear medical imaging, Scanning Electron Microscopy (SEM)</td>
</tr>
</tbody>
</table>
Mathematics of Imaging

- **Forward model** - The field $U = U(x, y, z)$ typically obeys a PDE dictated by the physics propagation in the medium

\[ \Theta\{U, \alpha\} = 0 \]

- The position-dependent quantity $\alpha = \alpha(x, y, z)$ – typically a coefficient in the PDE

- Example - Helmholtz equation

\[ \nabla^2 U(x) + k^2(x)U(x) = S(x, \omega) \]

- Typically – PDE is linear in $U$ and nonlinear in $\alpha$
Mathematics of Imaging

- Inhomogeneity causes perturbation in the medium properties
  \[ k^2(x) = k_0^2(x) + f(x) \]
  Image to be reconstructed
  Background medium
  Perturbed medium

- Model for the incident field
  \[ U_p(y) = \int G(y, x, \omega) S(x, \omega) dx \]

- Linearized model for the scattered field
  \[ U_s(z) = \int \int G(z, y, \omega) G(y, x, \omega) S(x, \omega) f(y) dx dy \]
Ray theoretic approximation to the Green’s function

\[ G(y, x, \omega) \approx e^{i\omega \tau(x, y)} \sum_{n=0}^{\infty} \frac{A_n(x, y)}{\omega^n} \]

\( \tau(x, y) \) is the solution of the Eikonal equation:

\[ |\nabla_x \tau(x, y)|^2 = k_0^2(x) \]

\( A_n(x, y) \) satisfy certain transport equations along rays connecting \( x \) and \( y \). Can be solved successively.
Mathematics of Imaging

- **Input-output relationship**

\[ u_s(z, t) = \mathcal{F}[f](z, t) \]

- **Kernel of \( \mathcal{F} \) is in the following form:**

\[ F(t, z, y) = \int e^{i\varphi(\omega, z, y, t)} A(\omega, z, y, t) d\omega \]

- \( \mathcal{F} \) is a Fourier Integral Operator (FIO) under some conditions
• FIO defines a Generalized Radon Transform

\[ u_s(z, t) = \mathcal{F}[f](z, t) \]

\[ F(t, z, y) = \int e^{i\varphi(\omega, z, y, t)} A(\omega, z, y, t) d\omega \]

• Kernel of \( \mathcal{F} \) shows \( \rightarrow \) weighted/filtered integral of an unknown function \( f \) along \text{curved manifolds} such as hyperplanes, lines, circles, ellipses, hyperbolas etc.

• Close connection between wave-based imaging and inverse problems of integral geometry
X-ray Computed Tomography

- Determine X-ray attenuation coefficient $f(x)$ along lines
- Measurements modeled as integrals along lines

$$\mathcal{F}_{X-ray}[f](L) = \int_{L} f(x) \, dx$$

- Manifold – Line
- Weight – 1
Emission Tomography

- A nuclear medicine imaging technique
- Body injected with gamma-emitting radio isotope. Isotopes accumulate at tumor sites
- Determine the concentration of radiating source

\[
\mathcal{F}_{ET}[f](L) = \int_{L} f(x) \exp \left( - \int_{L(x)} \mu(y) \, dy \right) \, dx
\]

- Manifold – Lines
- Weight – Exponential
**Synthetic Aperture Imaging - Radar**

- Electromagnetic waves are transmitted along a moving antenna and scattered waves are measured along the same (or different) antenna.
- Determine the reflectivity of the ground.

\[
F_{SAR}[f](s, t) = \int e^{-i\omega(t-\frac{2}{c_0}|x-\gamma(s)|)} A(x, s, \omega) f(x) \, dx \, d\omega
\]

- Manifold – Circles (or ellipses)
- Weight – Defined by the antenna trajectory and transmitted waveforms.

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Similar to synthetic aperture radar

Synthetic aperture sonar uses acoustic waves instead

Transmitted signal attenuates due to absorption for the case of GPR and GPS

Manifold – Circles (or ellipses)

Weight – Defined by antenna/transducer trajectory, transmitted waveforms as well as ground absorption

Measurement model:

$$F_{BSAI}[f](t, y, z) = \int e^{-i\omega(t-(\tau(x,y)+\tau(x,z)))} A(x, y, z, \omega) f(x) dx d\omega$$
Thermoacoustic Tomography

- Short electromagnetic pulse sent in a biological tissue
- Energy absorbed by the object depends on density, concentration of oxygen and hemoglobin, water content etc.
- Absorbed energy triggers a thermoacoustic response
- Determine the absorption coefficient

\[ \mathcal{F}_{\text{TAT}}[f](x,t) = \int_{|y|=1} f(x + ty) \, d\sigma(y) \]

Manifold – Spheres
Weight – 1
Doppler Tomography

- Use acoustic or EM waves
- Determine the velocity of a moving fluid such as blood
- Model for the velocity vector field
  \[ f(x) = [f_1(x), f_2(x), f_3(x)]^T \]
- Doppler transform
  \[ \mathcal{F}_{Dop}[f](x, \theta) = \int \sum_{i=1}^{3} f_i(x + t\theta) \theta_i \, dt \]
- Manifold – Lines
- Weight – \( \theta_i \)

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Microlocal Analysis and Imaging

- FIO theory provides a common framework for a wide range of imaging problems
- Medium heterogeneities/ anomalies can be mathematically described by the concept of singularities and wavefront sets
- Microlocal analysis tells us how the singularities of the medium propagate to the measured data
- Recovering the heterogeneities involve back-propagating the singularities in the data to the image domain via inversion of FIOs
Advantages of Microlocal Inversion

• Leads to new and novel imaging modalities
• Reconstructs images with correct “geometry”
• Applicable under non-ideal imaging scenarios
  – Incomplete or limited view data
  – Noise and clutter
  – Arbitrary source and detector locations
  – Complex environments involving multiple scattering
• Computational efficiency
Conclusion

Diverse Problems

Sensing & imaging

CT  SPECT  TAT  Sonar  Ultrasound tomography  Doppler tomography  Radar imaging  Seismic imaging

Inverse problems in integral geometry

Common Approach

Microlocal Analysis